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Intrinsic algebraic entropy

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Abstract

The new notion of intrinsic algebraic entropy $\widetilde{\text{ent}}$ of endomorphisms of Abelian groups is introduced and investigated. The intrinsic algebraic entropy is compared with the algebraic entropy, a well-known numerical invariant introduced in the sixties and recently deeply studied also in its relations to other fields of Mathematics. In particular, it is shown that the intrinsic algebraic entropy and the algebraic entropy coincide on endomorphisms of torsion Abelian groups, and their precise relation is clarified in the torsion-free case. The Addition Theorem and the Uniqueness Theorem are also proved for $\widetilde{\text{ent}}$, in analogy with similar results for the algebraic entropy. Furthermore, a relevant connection of $\widetilde{\text{ent}}$ to the algebraic entropy of a continuous endomorphism of a locally compact Abelian group G is pointed out; this allows for the computation of the algebraic entropy in case G is totally disconnected.

1 Introduction

The algebraic entropy ent for endomorphisms of Abelian groups, as sketched in [2] and developed in [25] by Weiss, and more recently in [9], is defined by using trajectories of finite subgroups, and so its natural setting is that of torsion Abelian groups. Peters gave a different definition of algebraic entropy in [16] for automorphisms of Abelian groups using finite subsets instead of finite subgroups; he obtained in this way the new invariant h which is non-trivial also for torsion-free Abelian groups. This notion was extended to endomorphisms of Abelian groups in [6], where it was proved that h coincides with the algebraic entropy ent for endomorphisms of torsion Abelian groups. Imitating [17] and [6], in [24] the definition of algebraic entropy was extended to all continuous endomorphisms of locally compact Abelian (LCA) groups (see §4.1). As noted in [7] also the commutativity of the groups can be removed in all these cases.

In this paper we extend the family of finite subgroups of an Abelian group G by a suitable larger family of subgroups depending on the fixed endomorphism ϕ of G , deriving a new notion of algebraic entropy of ϕ (see Definition 1.2). This suitable notion of subgroup (see Definition 1.1) is inspired by that of inert subgroup introduced by Belyaev in [3] for inner automorphisms of non-commutative groups and investigated in that setting in [4], [18] and [5]. (Recall that a subgroup H of G is *inert* if $[H : H \cap \phi(H)]$ is finite for every inner automorphism ϕ of G ; Belyaev [3] gives credit to Kegel for coining the term “inert subgroup”.)

Following [6], the definition of algebraic entropy $h(\phi)$ of an endomorphism $\phi : G \rightarrow G$ of an Abelian group G makes use of finite subsets F of G and their *partial ϕ -trajectories*

$$T_n(\phi, F) = F + \phi(F) + \phi^2(F) + \dots + \phi^{n-1}(F)$$

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for all $n \in \mathbb{N}_+$, which are still finite subsets of G . Moreover, the ϕ -trajectory of F is

$$T(\phi, F) = \bigcup_{n \in \mathbb{N}_+} T_n(\phi, F).$$

If F is a finite subgroup of G , then $T(\phi, F)$ is the smallest ϕ -invariant subgroup of G containing F . According to [6], the limit

$$H(\phi, F) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, F)|}{n} \quad (1.1)$$

does exist and is finite, and it is called the *algebraic entropy with respect to F* of ϕ . The *algebraic entropy* of ϕ is

$$h(\phi) = \sup\{H(\phi, F) : F \subseteq G \text{ finite}\},$$

while

$$\text{ent}(\phi) = \sup\{H(\phi, F) : F \in \mathcal{F}(G)\},$$

where $\mathcal{F}(G)$ denotes the family of all finite subgroups of G . As previously noted, h and ent coincide when dealing with torsion Abelian groups. Indeed, denoting as usual by $t(G)$ the torsion subgroup of the Abelian group G (that is fully invariant), we have that $\text{ent}(\phi) = \text{ent}(\phi \upharpoonright_{t(G)})$ as every finite subgroup of G is contained in $t(G)$, and $h(\phi \upharpoonright_{t(G)}) = \text{ent}(\phi \upharpoonright_{t(G)})$ since every finite subset of $t(G)$ generates a finite subgroup containing it.

According to a basic fact in the framework of the algebraic entropy ent (see [9, Lemma 1.1]), for a finite subgroup F of G , the sequence of non-negative integers

$$\alpha_n = \frac{|T_{n+1}(\phi, F)|}{|T_n(\phi, F)|} \quad (1.2)$$

is non-increasing, hence it is stationary. A crucial consequence of this fact is that

$$H(\phi, F) = \lim_{n \rightarrow \infty} \log \alpha_n = \log \alpha, \quad (1.3)$$

where $\alpha = \alpha_n$ for all n large enough (see [9, Proposition 1.3]).

A natural question arises: can we compute $H(\phi, F)$ using the formula in (1.3), without asking finiteness of the subgroup F ? The idea of using (1.3) instead of (1.1) to compute the algebraic entropy of ϕ with respect to F triggered the new notion of entropy, that we call *intrinsic algebraic entropy*.

Since for all $n \in \mathbb{N}_+$ and for every subgroup H of G we have

$$\frac{T_{n+1}(\phi, H)}{T_n(\phi, H)} \cong \frac{T_{n+1}(\phi, H)/H}{T_n(\phi, H)/H}, \quad (1.4)$$

in order to get the limit in (1.3) we do not need finiteness of $T_n(\phi, H)$ for all $n \in \mathbb{N}_+$. What really matters is finiteness of $T_n(\phi, H)/H$ for all $n \in \mathbb{N}_+$, hence we are led to the following

Definition 1.1. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism. A subgroup H of G is ϕ -inert if $(H + \phi(H))/H$ is finite.*

Since $T_2(\phi, H)/H = (H + \phi(H))/H$ is finite if and only if $T_n(\phi, H)/H$ is finite for all $n \in \mathbb{N}_+$ (see Lemma 2.1), in view of (1.4) the limit in (1.3) exists and is finite for a ϕ -inert subgroup H of G (see Lemma 3.2); hence we can set

$$\widetilde{\text{ent}}(\phi, H) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, H)/H|}{n}.$$

Taking the supremum of these non-negative real numbers, ranging H in the set $\mathcal{I}_\phi(G)$ of the ϕ -inert subgroups of G , we have the announced new notion:

Definition 1.2. Let G be an Abelian group and $\phi : G \rightarrow G$ an endomorphism. The intrinsic algebraic entropy of ϕ is

$$\widetilde{\text{ent}}(\phi) = \sup_{H \in \mathcal{I}_\phi(G)} \widetilde{\text{ent}}(\phi, H).$$

In Section 2 we study the family $\mathcal{I}_\phi(G)$ of the ϕ -inert subgroups of an Abelian group G , where ϕ is an endomorphism of G . In particular, we show that $\mathcal{I}_\phi(G)$ is a bounded sublattice of the lattice of all the subgroups of G (see Lemma 2.6), containing all finite subgroups, all finite-index subgroups, as well as all ϕ -invariant and fully invariant subgroups of G ; but in general $\mathcal{I}_\phi(G)$ is wider.

In Section 3 we give the basic properties of the intrinsic algebraic entropy; some of them are similar to those of the algebraic entropy, but there are also important differences. For every endomorphism ϕ of any Abelian group G , these entropies are related by the inequalities $\text{ent}(\phi) \leq \widetilde{\text{ent}}(\phi) \leq h(\phi)$; as for endomorphisms of torsion Abelian groups the equality $\text{ent}(\phi) = h(\phi)$ holds, in this case this common value coincides also with $\widetilde{\text{ent}}(\phi)$. Therefore, the real novelty of intrinsic algebraic entropy is confined to the class of non-torsion Abelian groups.

The topological setting provides another motivation for introducing ϕ -inert subgroups (see §4.1). Let G be a locally compact Abelian group and $\mathcal{B}(G)$ be the family of all compact open subgroups of G . Then every $U \in \mathcal{B}(G)$ is ϕ -inert for every continuous endomorphism $\phi : G \rightarrow G$ (i.e., $\mathcal{B}(G) \subseteq \mathcal{I}_\phi(G)$) and $\widetilde{\text{ent}}(\phi, U)$ coincides with the algebraic entropy of ϕ with respect to U (see Theorem 4.3). In particular, $\widetilde{\text{ent}}(\phi)$ is an upper bound for the algebraic entropy of ϕ whenever G is totally disconnected (see Remark 4.4). On the other hand, if G is totally disconnected and ϕ is a topological automorphism of G , then the finite index $[U + \phi(U) : U] = [\phi(U) : U \cap \phi(U)]$, for $U \in \mathcal{B}(G)$, is involved also in the definition of the *scale function* $s(\phi)$ of Willis [26, 27] given by $s(\phi) = \min\{[\phi(U) : U \cap \phi(U)] : U \in \mathcal{B}(G)\}$.

In the rest of Section 4 we compute the intrinsic algebraic entropy of endomorphisms of finite dimensional \mathbb{Q} -vector spaces and we reveal its relation to the so-called Algebraic Yuzvinski Formula recently proved in [12]. The Algebraic Yuzvinski Formula states that, for an endomorphism $\phi : \mathbb{Q}^m \rightarrow \mathbb{Q}^m$ with $m \in \mathbb{N}_+$, the value of the algebraic entropy can be computed as

$$h(\phi) = \log s + \sum_{|\lambda_i| > 1} \log |\lambda_i|,$$

where $\{\lambda_1, \dots, \lambda_m\} \subseteq \mathbb{C}$ are the eigenvalues of ϕ and s is the least common multiple of the denominators of the coefficients of the characteristic polynomial of ϕ . The term $\log s$ in this formula seems to be somewhat mysterious; now we see in Theorem 4.2 that

$$\widetilde{\text{ent}}(\phi) = \log s. \tag{1.5}$$

In Section 5 we carefully study the particular case when the group G is the ϕ -trajectory of a finitely generated subgroup, in other words, when G is a finitely generated $\mathbb{Z}[X]$ -module, where the action of an integer polynomial over G is induced by ϕ . The detailed description of this case allows us to prove the Addition Theorem for the intrinsic algebraic entropy, stating that $\widetilde{\text{ent}}$ is an additive invariant of the category $\text{Mod}(\mathbb{Z}[X])$. This result, together with the property of being upper continuous, makes the intrinsic algebraic entropy a length function on $\text{Mod}(\mathbb{Z}[X])$, similarly to the algebraic entropy h (see [6]).

In Section 6 we prove a Uniqueness Theorem, characterizing the intrinsic algebraic entropy in a similar fashion, *mutatis mutandis*, to the analogous theorems proved for the two above mentioned entropies ent and h in [9] and [6], respectively. Indeed, we see that $\widetilde{\text{ent}}$ is the unique length function on $\text{Mod}(\mathbb{Z}[X])$ coinciding with ent on torsion Abelian groups and satisfying (1.5).

In [5] an automorphism $\phi : G \rightarrow G$ of a group G is defined to be *inertial* if $[H : H \cap \phi(H)]$ and $[\phi(H) : H \cap \phi(H)]$ are finite for every subgroup H of G (note that in Definition 1.1 we consider endomorphisms of Abelian groups, not only automorphisms, and we require that $H \cap \phi(H)$ has finite index only in $\phi(H)$, and not in H). Inspired by this and similarly to what is done in [4] and [18] for inner automorphisms, one can introduce a smaller class of subgroups of an Abelian group G , letting ϕ ranging in the ring $\text{End}(G)$ of all endomorphisms of G . Namely, call a subgroup H of an Abelian group G *fully inert* if H is ϕ -inert for every endomorphism ϕ of G . The family $\bigcap_{\phi \in \text{End}(G)} \mathcal{I}_\phi(G)$ of all fully inert subgroups of G contains all finite subgroups, all finite-index subgroups and all fully invariant subgroups of G . Fully inert subgroups are investigated in the papers [8] and [10], with particular attention paid to the Abelian groups which are fully inert in divisible groups and free groups respectively.

Notation and terminology

We denote by \mathbb{N} , \mathbb{N}_+ , \mathbb{P} , \mathbb{Z} and \mathbb{Q} the set of the naturals, positive naturals, primes, integers and rationals respectively. For every $p \in \mathbb{P}$ we denote by \mathbb{J}_p the ring of p -adic integers, by \mathbb{Q}_p the field of p -adic numbers and by $|\cdot|_p$ the p -adic norm on \mathbb{Q}_p ; this is defined by $|p|_p = 1/p$ and $|q|_p = 1$ for every $q \in \mathbb{P}$ with $q \neq p$. Let \mathbb{R} be the set of real numbers, $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}$ and $\mathbb{R}^* = \mathbb{R}_{\geq 0} \cup \{\infty\}$.

For an Abelian group G we denote by $t(G)$ the torsion subgroup of G and by $\text{rk}(G) = \dim_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$ the torsion-free rank of G . Moreover, $D(G)$ denotes a divisible hull of G and, if G is torsion-free and L is a subgroup of G , L^* denotes the pure closure of L in G . We recall that $L^*/L = t(G/L)$ and so $G/L^* \cong (G/L)/(L^*/L) = (G/L)/t(G/L)$, hence we have the short exact sequence

$$0 \rightarrow L^* \rightarrow G \rightarrow (G/L)/t(G/L) \rightarrow 0. \quad (1.6)$$

For a ring R we denote by $\text{Mod}(R)$ the category of the left R -modules.

2 Basic properties of ϕ -inert subgroups

In this section we prove the basic facts concerning inert subgroups, that will be used in the investigation of the intrinsic algebraic entropy. We start from the next lemma collecting some useful properties of inert subgroups.

Lemma 2.1. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism and H a ϕ -inert subgroup of G . Then:*

- (a) $T_n(\phi, H)/H$ is finite for all $n \in \mathbb{N}_+$;
- (b) $T(\phi, H)/H$ is torsion;
- (c) if G is torsion-free, then $T(\phi, H)$ is contained in H^* and H^* is ϕ -invariant.

Proof. (a) We proceed by induction. For $n = 2$ finiteness is ensured by definition. Let us assume that $n \geq 2$ and $T_n(\phi, H)/H$ is finite. The map $T_n(\phi, H)/H \rightarrow T_{n+1}(\phi, H)/T_n(\phi, H)$ induced by ϕ is well-defined and surjective, therefore $T_{n+1}(\phi, H)/T_n(\phi, H)$ is finite. Since $T_{n+1}(\phi, H)/T_n(\phi, H)$ is isomorphic to $(T_{n+1}(\phi, H)/H)/(T_n(\phi, H)/H)$, $T_{n+1}(\phi, H)/H$ is finite as well, as an extension of $T_n(\phi, H)/H$ by this group.

(b) Since $T(\phi, H) = \bigcup_{n \in \mathbb{N}_+} T_n(\phi, H)$, it follows that

$$T(\phi, H)/H = \bigcup_{n \in \mathbb{N}_+} T_n(\phi, H)/H;$$

so $T(\phi, H)/H$, as a union of finite subgroups, is torsion.

(c) The first claim is an immediate consequence of item (b), as H^*/H is the torsion subgroup of G/H . To prove that H^* is ϕ -invariant, note that the ϕ -trajectory $T = T(\phi, H)$ is ϕ -invariant in G and H^*/T is torsion. If $x \in H^*$, then $kx \in T$ for a suitable integer $k \neq 0$. So $k\phi(x) = \phi(kx) \in T$, hence $\phi(x) \in H^*$, which is the pure closure of T in G , consequently H^* is ϕ -invariant in G . \square

The following characterization of ϕ -inert subgroups is needed later on.

Proposition 2.2. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism and H a subgroup of G . Then the following conditions are equivalent:*

- (a) H is ϕ -inert;
- (b) there exists a finite subset F of H such that $T_2(\phi, H) = H + \phi(F)$;
- (c) there exists a finite subset F of H such that $T_n(\phi, H) = H + T_n(\phi, F)$ for every $n \geq 2$.

If G is torsion, the subset F in (b) and (c) can be assumed to be a finite subgroup.

Proof. (a) \Rightarrow (b) Since $(H + \phi(H))/H$ is finite by hypothesis, there are finitely many cosets of $\phi(H)$ modulo H . Let F be a finite subset of H , such that $H + \phi(H) = H + \phi(F)$. If G is torsion and F denotes the subgroup generated by these cosets, we have the same equality.

(b) \Rightarrow (c) We induct on $n \in \mathbb{N}_+$. The hypothesis gives $H + \phi(H) = H + \phi(F)$ for a suitable finite subset F of H . Assume that $T_n(\phi, H) = H + T_n(\phi, F)$ for the same finite subset F , and for some $n \geq 2$. Then, using that $H + \phi(H) = H + \phi(F)$ and that $H = H + F$, so also $\phi(H) = \phi(H) + \phi(F)$, we get

$$\begin{aligned} T_{n+1}(\phi, H) &= H + \phi(T_n(\phi, H)) = H + \phi(H) + \phi(T_n(\phi, F)) = \\ &= H + \phi(F) + \phi(T_n(\phi, F)) = H + T_{n+1}(\phi, F). \end{aligned}$$

We can replace “finite subset” by “finite subgroup” if G is torsion.

(c) \Rightarrow (b) \Rightarrow (a) are trivial. \square

Another useful property of inert subgroups is the following one.

Lemma 2.3. *Let G be an Abelian group and $\phi : G \rightarrow G$ an endomorphism. A subgroup H of G is ϕ -inert if and only if $\overline{H} = (H + t(G))/t(G)$ is $\overline{\phi}$ -inert in $\overline{G} = G/t(G)$, where $\overline{\phi} : \overline{G} \rightarrow \overline{G}$ is the endomorphism induced by ϕ , and $((H + \phi(H))/H) \cap ((H + t(G))/H)$ is finite. The latter condition is automatically satisfied whenever H is finitely generated.*

Proof. Let us denote $T_n(\phi, H)$ by T_n for every $n \in \mathbb{N}_+$. The canonical surjection $T_n \rightarrow (T_n + t(G))/t(G)$ induces the exact sequence

$$0 \rightarrow (T_n \cap (H + t(G)))/H \rightarrow T_n/H \rightarrow (T_n + t(G))/(H + t(G)) \rightarrow 0.$$

By Lemma 2.1, H is ϕ -inert if and only if T_n/H is finite; in view of the above exact sequence, this is equivalent to the facts that $(T_n + t(G))/(H + t(G)) \cong ((T_n + t(G))/t(G))/((H + t(G))/t(G))$ is finite (that is, \overline{H} is $\overline{\phi}$ -inert) and $(T_n \cap (H + t(G)))/H$ is finite. For $n = 2$ this amounts to say that $((H + \phi(H))/H) \cap ((H + t(G))/H)$ is finite; conversely, if this is true, then the same inductive argument used in Lemma 2.1 shows that $(T_n \cap (H + t(G)))/H$ is finite.

The last claim follows from the fact that, if H is finitely generated, then $(H + \phi(H))/H$ and its subgroups are finitely generated. So it remains to add that a finitely generated torsion Abelian group is finite. \square

The next lemma will be useful in proving Theorem 4.2.

Lemma 2.4. *Let G be a torsion-free Abelian group of finite rank and $\phi : G \rightarrow G$ an endomorphism. Then every finitely generated subgroup H of G of maximal rank is ϕ -inert.*

Proof. The quotient $(H + \phi(H))/H$ is torsion, being a subgroup of G/H , and it is finitely generated being isomorphic to the quotient $\phi(H)/(H \cap \phi(H))$ of $\phi(H)$. Hence $(H + \phi(H))/H$ is finite, as desired. \square

The hypothesis that H is finitely generated in Lemma 2.4 cannot be removed, as shown in [8, Example 2.5]. Obviously, Lemma 2.4 fails also if we do not assume that the finitely generated subgroup H of G has maximal rank; in fact, if $\text{rk}(H) < \text{rk}(G)$, it can happen that $\phi(H)$ is not contained in the pure closure H^* of H , and this implies that H is not ϕ -inert.

The next proposition partially reverses item (c) of Lemma 2.1 for finitely generated subgroups.

Proposition 2.5. *Let G be a torsion-free Abelian group, $\phi : G \rightarrow G$ an endomorphism and H a finitely generated subgroup of G . Then H is ϕ -inert if and only if H^* is ϕ -invariant.*

Proof. By Lemma 2.1(c), we need only to prove that if H^* is ϕ -invariant then H is ϕ -inert. As H and H^* have the same rank, Lemma 2.4 implies that H is ϕ -inert in H^* , hence it is also ϕ -inert in G . \square

The following lemma shows that $\mathcal{I}_\phi(G)$ is a sublattice of the lattice of all the subgroups of G (this lattice is bounded, as 0 and G are always ϕ -inert).

Lemma 2.6. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism, H and H' two ϕ -inert subgroups of G . Then $H \cap H'$ and $H + H'$ are both ϕ -inert.*

Proof. The embeddings

$$\begin{aligned} \frac{(H \cap H') + \phi(H \cap H')}{H \cap H'} &\rightarrow \frac{(H \cap H') + \phi(H \cap H') + H}{H} \oplus \frac{(H \cap H') + \phi(H \cap H') + H'}{H'} \\ &\subseteq \frac{H + \phi(H)}{H} \oplus \frac{H' + \phi(H')}{H'}. \end{aligned}$$

imply that $H \cap H'$ is ϕ -inert. Concerning $H + H'$, we note that

$$\frac{H + H' + \phi(H + H')}{H + H'} = \frac{H + \phi(H) + H'}{H + H'} + \frac{H' + \phi(H') + H}{H + H'}$$

is a quotient of the finite group $((H + \phi(H))/H) \oplus ((H' + \phi(H'))/H')$, hence also $H + H'$ is ϕ -inert. \square

A last useful property of ϕ -inert subgroups is contained in the next lemma.

Lemma 2.7. *Let G be an Abelian group and $\phi : G \rightarrow G$ an endomorphism. If H is ϕ -inert, then for every $n \in \mathbb{N}_+$:*

- (a) $\phi^n(H)$ is ϕ -inert;
- (b) H is ϕ^n -inert.

Proof. Let $n \in \mathbb{N}_+$. To prove (a) it suffices to note that the homomorphism $H + \phi(H)/H \rightarrow \phi^n(H) + \phi^{n+1}(H)/\phi^n(H)$ induced by ϕ^n is surjective. To verify (b) observe that $H + \phi^n(H) \subseteq T_n(\phi, H)$; since $T_n(\phi, H)/H$ is finite by Lemma 2.1, also $H + \phi^n(H)/H$ is finite. \square

Lemma 2.7(b) shows that for an endomorphism $\phi : G \rightarrow G$ of an Abelian group G , a subgroup H of G is ϕ -inert if and only if H is ϕ^n -inert for every $n \in \mathbb{N}_+$. In other words,

$$\mathcal{I}_\phi(G) = \bigcap_{n \in \mathbb{N}_+} \mathcal{I}_{\phi^n}(G).$$

If $\phi : G \rightarrow G$ is an automorphism, then $\mathcal{I}_\phi(G) \neq \mathcal{I}_{\phi^{-1}}(G)$ may occur (see Example 3.1(b)).

3 The intrinsic algebraic entropy and its properties

We start this section introducing the following basic example concerning the Bernoulli shifts.

Example 3.1. For an arbitrary Abelian group define the right Bernoulli shift $\beta_G : G^{(\mathbb{N})} \rightarrow G^{(\mathbb{N})}$ by $\beta_G(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$, and the two-sided (right) Bernoulli shift $\sigma_G : G^{(\mathbb{Z})} \rightarrow G^{(\mathbb{Z})}$ by $\sigma_G((x_n)_{n \in \mathbb{Z}}) = (x_{n-1})_{n \in \mathbb{Z}}$. Note that σ_G is an automorphism, while β_G is a non-surjective monomorphism.

- (a) For p a prime, for the two-sided Bernoulli shift $\sigma_{\mathbb{Z}(p)} : \mathbb{Z}(p)^{(\mathbb{Z})} \rightarrow \mathbb{Z}(p)^{(\mathbb{Z})}$ the subgroup $H = \bigoplus_{-\infty}^0 \mathbb{Z}(p)$ is infinite and $\sigma_{\mathbb{Z}(p)}$ -inert.
- (b) For the two-sided Bernoulli shift $\sigma_{\mathbb{Z}} : \mathbb{Z}^{(\mathbb{Z})} \rightarrow \mathbb{Z}^{(\mathbb{Z})}$ the subgroup $H = \bigoplus_0^\infty \mathbb{Z}$ is $\sigma_{\mathbb{Z}}$ -invariant, hence $\sigma_{\mathbb{Z}}$ -inert, but it is not $\sigma_{\mathbb{Z}}^{-1}$ -inert.

3.1 Examples and comparison with the algebraic entropy

The following lemma shows that the limit defining the intrinsic algebraic entropy exists and it is finite.

Lemma 3.2. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism and H a ϕ -inert subgroup of G . Then the limit*

$$\lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, H)/H|}{n},$$

exists and it is finite.

Proof. In order to simplify notation, we set $T_n = T_n(\phi, H)$ for each $n \in \mathbb{N}_+$. Being H a ϕ -inert subgroup of G , T_n/H is finite for each $n \in \mathbb{N}_+$ by Lemma 2.1(a). Set

$$\alpha_{n+1} = \frac{|T_{n+1}/H|}{|T_n/H|} = \left| \frac{T_{n+1}}{T_n} \right| = \left| \frac{\phi^n(H)}{T_n \cap \phi^n(H)} \right|.$$

For each $n \geq 2$, the group $\phi^n(H)/(T_n \cap \phi^n(H))$ is a quotient of the group

$$B_n = \phi^n(H)/(\phi(T_{n-1}) \cap \phi^n(H)) \cong \phi(T_n)/\phi(T_{n-1}) \cong T_n/(T_{n-1} + (T_n \cap \ker \phi)).$$

Therefore α_{n+1} divides $\beta_n = |B_n|$. On the other hand, since the latter group is a quotient of $\frac{T_n/H}{T_{n-1}/H}$, we get that β_n divides α_n so, in particular, α_{n+1} divides α_n . We conclude that the sequence of non-negative integers $(\alpha_n)_{n \in \mathbb{N}_+}$ is non-increasing, hence it is stationary. So, as in case of H finite (see [9, Proposition 1.3]), it follows that

$$\lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, H)/H|}{n} = \log \alpha,$$

where $\alpha = \alpha_n$ for all n large enough. □

Now we give a characterization of the ϕ -inert subgroups H with $\widetilde{\text{ent}}(\phi, H) = 0$:

Proposition 3.3. *Let $\phi : G \rightarrow G$ be an endomorphism of an Abelian group G and H a ϕ -inert subgroup of G . The following conditions are equivalent:*

- (a) $\widetilde{\text{ent}}(\phi, H) = 0$;
- (b) H has finite index in its ϕ -trajectory $T(\phi, H)$.

Consequently $\widetilde{\text{ent}}(\phi) = 0$ if and only if every ϕ -inert subgroup of G has finite index in its ϕ -trajectory. In particular, $\widetilde{\text{ent}}(\phi) = 0$ if all subgroups of G are ϕ -invariant.

Proof. The proof of Lemma 3.2 shows that $\widetilde{\text{ent}}(\phi, H) = \log \alpha_n$ for all $n \in \mathbb{N}_+$ large enough, where $\alpha_n = |T_{n+1}(\phi, H)/T_n(\phi, H)|$. As $T(\phi, H) = \bigcup_{n \in \mathbb{N}_+} T_n(\phi, H)$, $\widetilde{\text{ent}}(\phi, H) = 0$ exactly if $T(\phi, H) = T_n(\phi, H)$ for some $n \in \mathbb{N}_+$. Since H has finite index in $T_n(\phi, H)$ by Lemma 2.1(a), the equivalence of (a) and (b) follows. The last claim is an immediate consequence. \square

Proposition 3.3 shows in particular that $\widetilde{\text{ent}}(\phi, H) = 0$ for every subgroup H of finite index in G .

We examine the basic examples of the endomorphisms of rational groups. Here and in the sequel we make use of the well known fact that, given a torsion-free Abelian group H of finite rank n , for each $b \in \mathbb{N}_+$ the cardinality of H/bH is at most b^n ([1, Theorem 0.1]); in particular, H/bH is finite.

Example 3.4. (1) Let $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ be a non-zero endomorphism of the group of the integers. Trivially, \mathbb{Z} has no non-trivial finite subgroups, hence $\text{ent}(\phi) = 0$. As ϕ acts as the multiplication by a non-zero integer, every subgroup of \mathbb{Z} is ϕ -invariant, so Proposition 3.3 implies that $\widetilde{\text{ent}}(\phi) = 0$.

(2) Let $\phi : \mathbb{Q} \rightarrow \mathbb{Q}$ be a non-zero endomorphism of the group of the rational numbers, and note that \mathbb{Q} has no non-trivial finite subgroups, hence, as before, $\text{ent}(\phi) = 0$. As ϕ acts as the multiplication by a non-zero fraction a/b , with $b > 0$ and a and b relatively prime, a non-zero subgroup H of \mathbb{Q} is ϕ -invariant exactly if $aH \leq bH$. This inequality is equivalent to the equality $H = bH$; in fact, $H = bH$ trivially implies $aH \leq bH$, and to see that the converse holds recall that $(a, b) = 1$ implies $aH + bH = H$. Hence H is ϕ -invariant exactly if it is p -divisible by all primes p dividing b . Furthermore, if H is an arbitrary subgroup of \mathbb{Q} , we have:

$$\frac{H + \phi(H)}{H} = \frac{H + ab^{-1}H}{H} \cong \frac{bH + aH}{bH} = \frac{H}{bH};$$

hence H is ϕ -inert, since H/bH is finite. Setting $m = |H/bH|$, by the fact quoted above we know that $m \leq b$; it is easy to show that $\widetilde{\text{ent}}(\phi, H) = \log m \leq \log b$ and consequently $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi, \mathbb{Z}) = \log b$. This example is extended to higher dimensions in the next Theorem 4.2. Recall that $h(\phi) = \max\{\log |a|, \log b\}$ (see [6] for a direct computation, otherwise one can apply the Algebraic Yuzvinski Formula proved in [12]). Hence, $h(\phi) > \widetilde{\text{ent}}(\phi)$ in case $|a| > b$.

(3) Generalizing the two preceding cases, let $\phi : G \rightarrow G$ be a non-zero endomorphism of a fixed subgroup G of the rational numbers. We can assume, without loss of generality, that $\mathbb{Z} \subseteq G$. As ϕ acts as the multiplication by a non-zero fraction a/b (with $(a, b) = 1$) belonging to the maximal subring contained in G , the same argument used in case (2) shows that $\widetilde{\text{ent}}(\phi) = \log b$.

An easy property of the intrinsic algebraic entropy is that $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(-\phi)$. On the other hand, $\widetilde{\text{ent}}$ is not preserved by taking the inverse of a given automorphism, as underlined in the following remark.

Remark 3.5. In case (2) of the preceding example, the automorphism ϕ^{-1} of \mathbb{Q} acts as the multiplication by b/a , hence $\widetilde{\text{ent}}(\phi^{-1}) = \log |a|$. This shows that the equalities $\text{ent}(\phi) = \text{ent}(\phi^{-1})$ and $h(\phi) = h(\phi^{-1})$, which hold for an automorphism ϕ , do not extend from the algebraic entropies to the intrinsic algebraic entropy.

We would like to compare the intrinsic algebraic entropy $\widetilde{\text{ent}}$ with the algebraic entropy ent and with the algebraic entropy h . In contrast with the inequality $h(\phi) > \widetilde{\text{ent}}(\phi)$ established in Example 3.4(2), we see now that always $h(\phi) = \widetilde{\text{ent}}(\phi)$ in case the Abelian group G is torsion.

Proposition 3.6. *Let G be an Abelian group and $\phi : G \rightarrow G$ an endomorphism. Then $\text{ent}(\phi) \leq \widetilde{\text{ent}}(\phi) \leq h(\phi)$. If G is torsion, then $\text{ent}(\phi) = \widetilde{\text{ent}}(\phi) = h(\phi)$.*

Proof. Since the family $\mathcal{I}_\phi(G)$ of the ϕ -inert subgroups of G contains the family $\mathcal{F}(G)$ of the finite subgroups of G , one obviously derives the inequality $\text{ent}(\phi) \leq \widetilde{\text{ent}}(\phi)$. Let H be a ϕ -inert subgroup. According to Proposition 2.2, there exists a finite subset F of H such that $T_n(\phi, H) = H + T_n(\phi, F)$ for all $n \in \mathbb{N}_+$. Then $|T_n(\phi, H)/H| = |(H + T_n(\phi, F))/H|$. Taking logarithms and dividing by n , the limit on the left hand side gives $\text{ent}(\phi, H)$. On the other hand, $|H + T_n(\phi, F)/H| \leq |T_n(\phi, F)|$, thus $\log |H + T_n(\phi, F)/H| \leq \log |T_n(\phi, F)|$ and after dividing by n we get $\widetilde{\text{ent}}(\phi, H) \leq \lim_{n \rightarrow \infty} \log |T_n(\phi, F)|/n \leq h(\phi)$, hence $\widetilde{\text{ent}}(\phi) \leq h(\phi)$. If G is torsion, it is well known (see [6] or [20]) that $\text{ent}(\phi) = h(\phi)$, hence the latter claim follows. \square

In Theorem 4.3 and Remark 4.4 we find a counterpart of the equality $\text{ent}(\phi) = \widetilde{\text{ent}}(\phi)$ in the case of a continuous endomorphism ϕ of a locally compact totally disconnected Abelian group G covered by compact subgroups (note that if such a group G is discrete, then it is torsion, so we obtain the case considered above).

Proposition 3.6 shows that for torsion Abelian groups it suffices to compute the intrinsic algebraic entropy with respect to finite subgroups, even if they may be less than the inert ones, as Example 3.1 shows.

As the Bernoulli shifts are crucial examples of endomorphisms in the theory of algebraic entropy, it is convenient to examine what happens also for their intrinsic algebraic entropy.

Example 3.7. We prove that $\widetilde{\text{ent}}(\beta_G) = \log |G|$ for the right Bernoulli shift $\beta_G : G^{(\mathbb{N})} \rightarrow G^{(\mathbb{N})}$ (see Example 3.1), where we intend that $\log |G| = \infty$ in case G is infinite. If G is torsion, this follows from Proposition 3.6 and the equality $\text{ent}(\beta_G) = \log |G|$ proved in [9, Example 1.9].

Now assume that G is a non-torsion Abelian group and choose a non-torsion element $g \in G$. Consider the element $x = (g, 0, 0, \dots) \in G^{(\mathbb{N})}$ and fix an integer $k > 1$. Let the subgroup H_k of $G^{(\mathbb{N})}$ be defined as follows:

$$H_k = x\mathbb{Z} + T(\beta_G, kx\mathbb{Z}) = g\mathbb{Z} \oplus \bigoplus_{i \geq 1} k\beta_G^i(x)\mathbb{Z}.$$

Then, for every $n \in \mathbb{N}_+$, we get:

$$T_n(\beta_G, H_k) = \bigoplus_{i < n} \beta_G^i(x)\mathbb{Z} \oplus \bigoplus_{i \geq n} k\beta_G^i(x)\mathbb{Z}.$$

It follows that $T_n(\beta_G, H_k)/H_k$ is isomorphic to the direct sum of $n - 1$ many copies of $\mathbb{Z}/k\mathbb{Z}$, so it has size k^{n-1} . Thus, every subgroup H_k is β_G -inert, $\widetilde{\text{ent}}(\beta_G, H_k) = \log k$, and consequently $\widetilde{\text{ent}}(\beta_G) = \infty$.

As in the case of the algebraic entropy ent , the range of the intrinsic algebraic entropy is a discrete subset of the real numbers; in fact, if it is finite, it is reached by a suitable inert subgroup as a maximum:

Lemma 3.8. *Let G be an Abelian group and $\phi : G \rightarrow G$ an endomorphism. If $\widetilde{\text{ent}}(\phi) < \infty$, then there exists a ϕ -inert subgroup H of G such that $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi, H) = \log n$ for some $n \in \mathbb{N}_+$.*

Proof. By the proof of Lemma 3.2, for every $H \in \mathcal{I}_\phi(G)$ there exists a positive integer n_H such that $\widetilde{\text{ent}}(\phi, H) = \log n_H$. Consider the set $\mathcal{F} = \{\widetilde{\text{ent}}(\phi, H) : H \in \mathcal{I}_\phi(G)\} \subseteq \{\log n : n \in \mathbb{N}_+\}$, and notice that \mathcal{F} is bounded by the finite value $\widetilde{\text{ent}}(\phi)$. One can easily conclude that \mathcal{F} is finite and then $\widetilde{\text{ent}}(\phi) = \sup_{H \in \mathcal{I}_\phi(G)} \log n_H = \max \mathcal{F}$ is an element of \mathcal{F} . \square

A consequence of the above lemma is that the set of finite values of $\widetilde{\text{ent}}$ (considered as a function from the class of all the endomorphisms of abelian groups) is contained in the subset $\{\log n : n \in \mathbb{N}_+\}$ of $\mathbb{R}_{\geq 0}$. Moreover, these two sets actually coincide, since by Example 3.7 we know that $\widetilde{\text{ent}}(\beta_G) = \log |G|$ if G is a finite Abelian group.

3.2 Typical properties of entropy functions

In the rest of this section we provide some fundamental properties of the intrinsic algebraic entropy that will allow us to prove the Uniqueness Theorem in Section 6. Similar results hold for the algebraic entropies ent and h (see [25], [9] and [6]); however, in the present situation, *ad hoc* arguments are needed.

In the next lemma we establish invariance of the intrinsic algebraic entropy under conjugation.

Lemma 3.9. *Let $\phi : G \rightarrow G$ be an endomorphism and $\xi : G \rightarrow G'$ an isomorphism of Abelian groups. Then $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\xi\phi\xi^{-1})$.*

Proof. Set $\psi = \xi\phi\xi^{-1}$. A subgroup H of G is ϕ -inert if and only if $\xi(H)$ is ψ -inert in G' , and an easy computation shows that $|T_n(\phi, H)/H| = |T_n(\psi, \xi(H))/\xi(H)|$ for every $n \in \mathbb{N}_+$. Therefore $\widetilde{\text{ent}}(\xi, H) = \widetilde{\text{ent}}(\psi, \xi(H))$ and the desired equality follows. \square

The next lemma establishes monotonicity of the intrinsic algebraic entropy under taking restrictions and quotients with respect to a ϕ -invariant subgroup.

Lemma 3.10. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism and H a ϕ -invariant subgroup of G . Then:*

- (a) $\widetilde{\text{ent}}(\phi) \geq \widetilde{\text{ent}}(\phi|_H)$ and $\widetilde{\text{ent}}(\phi) \geq \widetilde{\text{ent}}(\bar{\phi})$, where $\bar{\phi} : G/H \rightarrow G/H$ is the endomorphism induced by ϕ ;
- (b) if K is a ϕ -inert subgroup of G and $\bar{K} = (K + H)/H$, then $K \cap H$ is $\phi|_H$ -inert, \bar{K} is $\bar{\phi}$ -inert and

$$\widetilde{\text{ent}}(\phi, K) \geq \widetilde{\text{ent}}(\phi|_H, H \cap K) + \widetilde{\text{ent}}(\bar{\phi}, \bar{K}).$$

Proof. (a) The first inequality is obvious, since a $\phi|_H$ -inert subgroup of H is also a ϕ -inert subgroup of G . Let now $\bar{K} = K/H$ be a $\bar{\phi}$ -inert subgroup of G/H , with $H \leq K \leq G$. Since

$$\frac{T_n(\bar{\phi}, \bar{K})}{\bar{K}} \cong \frac{T_n(\phi, K)}{K}$$

for all $n \in \mathbb{N}_+$, K is ϕ -inert in G and $\widetilde{\text{ent}}(\bar{\phi}, \bar{K}) = \widetilde{\text{ent}}(\phi, K) \leq \widetilde{\text{ent}}(\phi)$. Hence the latter inequality also holds.

(b) It is easy to check that the following short sequence is exact for all $n \geq 1$:

$$0 \rightarrow \frac{T_n(\phi, K) \cap (K + H)}{K} \rightarrow \frac{T_n(\phi, K)}{K} \rightarrow \frac{T_n(\phi, K) + H}{K + H} \rightarrow 0.$$

The last non-zero term in the above sequence is isomorphic to $T_n(\bar{\phi}, \bar{K})/\bar{K}$, because $(T_n(\phi, K) + H)/H = T_n(\bar{\phi}, \bar{K})$. The first non-zero term contains the subgroup

$$\frac{(K + T_n(\phi|_H, H \cap K)) \cap (K + H)}{K} \cong \frac{T_n(\phi|_H, H \cap K)}{(H \cap K)}.$$

Consequently we deduce that

$$|T_n(\phi, K)/K| \geq |T_n(\bar{\phi}, \bar{K})/\bar{K}| \cdot |T_n(\phi|_H, H \cap K)/(H \cap K)|.$$

Taking logarithms, dividing by n and making n tend to ∞ , we get the desired inequality. \square

We apply the next lemma in the proof of Lemma 3.12 below.

Lemma 3.11. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism and H a subgroup of G .*

- (a) *If H is ϕ -inert and $H' = T_k(\phi, H)$ for some $k \in \mathbb{N}_+$, then H' is ϕ -inert and $\widetilde{\text{ent}}(\phi, H) = \widetilde{\text{ent}}(\phi, H')$.*
- (b) *If H is ϕ^k -inert for some $k \in \mathbb{N}_+$ and $H' = T_k(\phi, H)$, then H' is ϕ -inert (so, in particular ϕ^k -inert) and $\widetilde{\text{ent}}(\phi^k, H) = \widetilde{\text{ent}}(\phi^k, H')$.*

Proof. (a) Since $H' + \phi(H') = T_{k+1}(\phi, H)$, we have $(H' + \phi(H'))/H' = T_{k+1}(\phi, H)/T_k(\phi, H)$. This quotient is finite, as it is a quotient of $T_{k+1}(\phi, H)/H$, which is finite by the assumption that H is ϕ -inert and Lemma 2.1(a). Moreover, since

$$\left| \frac{T_n(\phi, H')}{H'} \right| = \left| \frac{T_{k+n}(\phi, H)}{H'} \right| = \left| \frac{T_{k+n}(\phi, H)}{H} \right| \cdot |H'/H|,$$

and $|H'/H|$ is a finite constant by Lemma 2.1(a), it follows that

$$\widetilde{\text{ent}}(\phi, H') = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, H')/H'|}{n} = \lim_{n \rightarrow \infty} \frac{\log |T_{k+n}(\phi, H)/H|}{n} = \widetilde{\text{ent}}(\phi, H).$$

(b) To prove that H' is ϕ -inert, we need to verify that $(H' + \phi(H'))/H'$ is finite. Since $H' + \phi(H') = T_{k+1}(\phi, H) = H' + \phi^k(H)$, we have that $(H' + \phi(H'))/H' \cong \phi^k(H)/(H' \cap \phi^k(H))$. This is finite since it is a quotient of $\phi^k(H)/(H \cap \phi^k(H)) \cong (H + \phi^k(H))/H$, which is finite by the assumption that H is ϕ^k -inert. Hence, H' is ϕ -inert. By Lemma 2.7 the subgroup H' is also ϕ^k -inert. Item (a) gives that $\widetilde{\text{ent}}(\phi^k, H) = \widetilde{\text{ent}}(\phi^k, H')$. \square

The next lemma establishes a “logarithmic law” for the intrinsic algebraic entropy.

Lemma 3.12. *Let G be an Abelian group and $\phi : G \rightarrow G$ an endomorphism. Then $\widetilde{\text{ent}}(\phi^k) = k \cdot \widetilde{\text{ent}}(\phi)$ for every $k \in \mathbb{N}_+$.*

Proof. Fix $k \in \mathbb{N}_+$. First we prove the inequality $\widetilde{\text{ent}}(\phi^k) \leq k \cdot \widetilde{\text{ent}}(\phi)$. Let H be a ϕ^k -inert subgroup of G and $H' = T_k(\phi, H)$. Lemma 3.11(b) yields that H' is ϕ -inert, so also ϕ^k -inert, and $\widetilde{\text{ent}}(\phi^k, H) = \widetilde{\text{ent}}(\phi^k, H')$.

Let $n \in \mathbb{N}_+$. Then $T_n(\phi^k, H') \subseteq T_{kn-k+1}(\phi, H')$ and so

$$\begin{aligned} \widetilde{\text{ent}}(\phi^k, H') &= \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi^k, H')/H'|}{n} \leq \lim_{n \rightarrow \infty} \frac{\log |T_{kn-k+1}(\phi, H')/H'|}{n} \\ &= \lim_{n \rightarrow \infty} \frac{\log |T_{kn-k+1}(\phi, H')/H'|}{kn-k+1} \cdot \lim_{n \rightarrow \infty} \frac{kn-k+1}{n} \\ &= k \cdot \lim_{n \rightarrow \infty} \frac{\log |T_{kn-n+1}(\phi, H')/H'|}{kn-k+1} = k \cdot \widetilde{\text{ent}}(\phi, H'). \end{aligned}$$

Therefore, $\widetilde{\text{ent}}(\phi^k, H) = \widetilde{\text{ent}}(\phi^k, H') \leq k \cdot \widetilde{\text{ent}}(\phi, H')$. Since H' is ϕ -inert, we can conclude that $\widetilde{\text{ent}}(\phi^k) \leq k \cdot \widetilde{\text{ent}}(\phi)$.

To prove the converse inequality, let H be a ϕ -inert subgroup of G and $H_1 = T_k(\phi, H)$. Moreover, let $n \in \mathbb{N}_+$. By Lemma 3.11(a) the subgroup H_1 is ϕ -inert and H_1/H is finite. By Lemma 2.7 the subgroup H_1 is also ϕ^k -inert. Moreover, $T_n(\phi^k, H_1) = T_{kn}(\phi, H)$ and so $|T_n(\phi^k, H_1)/H_1| \cdot |H_1/H| = |T_{kn}(\phi, H)/H|$. Then, as $|H_1/H|$ is constant,

$$\frac{1}{k} \cdot \widetilde{\text{ent}}(\phi^k, H_1) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi^k, H_1)/H_1|}{kn} = \lim_{n \rightarrow \infty} \frac{\log |T_{kn}(\phi, H)/H|}{kn} = \widetilde{\text{ent}}(\phi, H).$$

We can conclude that $\widetilde{\text{ent}}(\phi^k) \geq k \cdot \widetilde{\text{ent}}(\phi)$. \square

Here we obtain a weaker form of the Addition Theorem with respect to direct sums (the general case is discussed in Section 5).

Lemma 3.13. *Let G_i be Abelian groups and $\phi_i : G_i \rightarrow G_i$ endomorphisms for $i = 1, 2$. Then $\widetilde{\text{ent}}(\phi_1 \oplus \phi_2) = \widetilde{\text{ent}}(\phi_1) + \widetilde{\text{ent}}(\phi_2)$.*

Proof. Let H_i be a ϕ_i -inert subgroup of G_i , for $i = 1, 2$. Setting $\phi = \phi_1 \oplus \phi_2$, for every $n \in \mathbb{N}_+$ we have $T_n(\phi, H_1 \oplus H_2)/(H_1 \oplus H_2) \cong (T_n(\phi_1, H_1)/H_1) \oplus (T_n(\phi_2, H_2)/H_2)$, therefore $H_1 \oplus H_2$ is ϕ -inert and

$$\widetilde{\text{ent}}(\phi, H_1 \oplus H_2) = \widetilde{\text{ent}}(\phi_1, H_1) + \widetilde{\text{ent}}(\phi_2, H_2). \quad (3.1)$$

Let H be a ϕ -inert subgroup of $G = G_1 \oplus G_2$, $p_i : G \rightarrow G_i$ the canonical projection and $H_i = p_i(H)$, for $i = 1, 2$. Each H_i is a ϕ_i -inert subgroup of G_i , as the homomorphism $(H + \phi(H))/H \rightarrow (H_i + \phi_i(H_i))/H_i$ induced by p_i is well-defined and surjective. Therefore $H_1 \oplus H_2$ is a ϕ -inert subgroup of G . Since for every $n \in \mathbb{N}_+$ the homomorphism $T_n(\phi, H)/H \rightarrow T_n(\phi, H_1 \oplus H_2)/(H_1 \oplus H_2)$ induced by $p_1 \oplus p_2$ is well-defined and injective, $\widetilde{\text{ent}}(\phi, H) \leq \widetilde{\text{ent}}(\phi, H_1 \oplus H_2)$. By (3.1) we can conclude that $\widetilde{\text{ent}}(\phi) \leq \widetilde{\text{ent}}(\phi_1) + \widetilde{\text{ent}}(\phi_2)$. If $\widetilde{\text{ent}}(\phi) = \infty$, by the latter inequality we can conclude that $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi_1) + \widetilde{\text{ent}}(\phi_2)$. So assume that $\widetilde{\text{ent}}(\phi)$ is finite. By Lemma 3.10(a) we have that $\widetilde{\text{ent}}(\phi_1)$ and $\widetilde{\text{ent}}(\phi_2)$ are finite as well. For $i = 1, 2$, let H_i be a ϕ_i -inert subgroup of G_i such that $\widetilde{\text{ent}}(\phi_i) = \widetilde{\text{ent}}(\phi_i, H_i)$. Then (3.1) gives

$$\widetilde{\text{ent}}(\phi) \geq \widetilde{\text{ent}}(\phi, H_1 \oplus H_2) = \widetilde{\text{ent}}(\phi_1, H_1) + \widetilde{\text{ent}}(\phi_2, H_2) = \widetilde{\text{ent}}(\phi_1) + \widetilde{\text{ent}}(\phi_2),$$

and this concludes the proof. \square

The next lemma shows that the intrinsic algebraic entropy is “continuous” with respect to direct limits.

Lemma 3.14. *Let G be an Abelian group and $\phi : G \rightarrow G$ an endomorphism, and assume that the group G is the direct limit of a family of ϕ -invariant subgroups $\{G_i : i \in I\}$. Let $\phi_i = \phi|_{G_i}$ for all $i \in I$. Then $\widetilde{\text{ent}}(\phi) = \sup_{i \in I} \widetilde{\text{ent}}(\phi_i)$.*

Proof. If H is a ϕ_i -inert subgroup of G_i , then it is also a ϕ -inert subgroup of G , therefore $\widetilde{\text{ent}}(\phi) \geq \widetilde{\text{ent}}(\phi_i)$ for all $i \in I$, hence $\widetilde{\text{ent}}(\phi) \geq \sup_{i \in I} \widetilde{\text{ent}}(\phi_i)$.

Conversely, let H be a ϕ -inert subgroup of G . By Lemma 3.10(b) $H \cap G_i$ is a ϕ_i -inert subgroup of G_i for every index $i \in I$. To conclude it is enough to prove that there exists an index $i \in I$ such that, for all $n \geq 1$,

$$|T_n(\phi, H)/H| \leq |T_n(\phi_i, H \cap G_i)/(H \cap G_i)|. \quad (3.2)$$

In fact, if this occurs, then $\widetilde{\text{ent}}(\phi, H) \leq \widetilde{\text{ent}}(\phi_i, H \cap G_i) \leq \widetilde{\text{ent}}(\phi_i)$, therefore $\widetilde{\text{ent}}(\phi, H) \leq \widetilde{\text{ent}}(\phi_i) \leq \sup_{i \in I} \widetilde{\text{ent}}(\phi_i)$, and hence $\widetilde{\text{ent}}(\phi) \leq \sup_{i \in I} \widetilde{\text{ent}}(\phi_i)$. By Proposition 2.2 there exists a finitely generated subset F of H such that for all $n \geq 1$

$$T_n(\phi, H) = H + T_n(\phi, F).$$

The subset F is contained in some G_i ; therefore, $F \subseteq H \cap G_i$ and $T_n(\phi, F) \subseteq T_n(\phi_i, H \cap G_i)$, so

$$(H \cap G_i) + T_n(\phi, F) \subseteq T_n(\phi_i, H \cap G_i).$$

Also the converse inclusion holds true; in fact, $T_n(\phi_i, H \cap G_i) \subseteq T_n(\phi, H) \cap G_i$, and since $T_n(\phi, F) \subseteq G_i$ we can conclude that $T_n(\phi, H) \cap G_i = (H + T_n(\phi, F)) \cap G_i = (H \cap G_i) + T_n(\phi, F)$. So we have the equality

$$T_n(\phi_i, H \cap G_i) = (H \cap G_i) + T_n(\phi, F).$$

Therefore $T_n(\phi, H)/H \cong (H + T_n(\phi, F))/H \cong T_n(\phi, F)/(H \cap T_n(\phi, F))$ is isomorphic to a quotient of

$$T_n(\phi_i, H \cap G_i)/(H \cap G_i) \cong ((H \cap G_i) + T_n(\phi, F))/(H \cap G_i) \cong T_n(\phi, F)/(H \cap G_i \cap T_n(\phi, F)),$$

hence the desired inequality in (3.2) is proved. \square

Thanks to Lemma 3.14 we can show now that, given an endomorphism $\phi : G \rightarrow G$ of a torsion-free Abelian group G , its intrinsic algebraic entropy equals that of its unique extension to the divisible hull of G .

Corollary 3.15. *Let G be a torsion-free Abelian group and ϕ an endomorphism of G . Let $\tilde{\phi} : D(G) \rightarrow D(G)$ be the unique extension of ϕ to the divisible hull. Then*

$$\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\tilde{\phi}).$$

Proof. We can identify G with a subgroup of $D(G)$, which is the direct limit of its $\tilde{\phi}$ -invariant subgroups of the form $\frac{1}{n!}G$, with n ranging in \mathbb{N}_+ . Hence, by Lemma 3.14, we have that $\widetilde{\text{ent}}(\tilde{\phi})$ is the supremum of the algebraic entropies of the restrictions of $\tilde{\phi}$ to each $\frac{1}{n!}G$. Furthermore, the groups $\frac{1}{n!}G$ are all isomorphic to G and the restriction of $\tilde{\phi}$ to $\frac{1}{n!}G$ is conjugated to ϕ through this isomorphism. Hence, $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\tilde{\phi} \upharpoonright_{(1/n!)G})$ for all $n \in \mathbb{N}_+$, and so $\widetilde{\text{ent}}(\tilde{\phi}) = \sup \left\{ \widetilde{\text{ent}}(\tilde{\phi} \upharpoonright_{(1/n!)G}) : n \in \mathbb{N}_+ \right\} = \widetilde{\text{ent}}(\phi)$. \square

The invariance of the intrinsic algebraic entropy under taking divisible hulls holds only for torsion-free Abelian groups. In fact, consider the torsion group $\mathbb{Z}(p)^{(\mathbb{N})}$, whose divisible hull is $\mathbb{Z}(p^\infty)^{(\mathbb{N})}$. Then $\widetilde{\text{ent}}(\beta_{\mathbb{Z}(p)}) = \text{ent}(\beta_{\mathbb{Z}(p)}) = \log p$ while $\widetilde{\text{ent}}(\beta_{\mathbb{Z}(p^\infty)}) = \infty$ (see Example 3.7).

Unlike the monotonicity of the function $F \mapsto H(\phi, F)$, its counterpart for $\widetilde{\text{ent}}$ fails, i.e., $\widetilde{\text{ent}}(\phi, H') \leq \widetilde{\text{ent}}(\phi, H)$ need not hold for ϕ -inert subgroups $H' \subseteq H$ of G (e.g., take $H = G$ and recall that $\widetilde{\text{ent}}(\phi, G) = 0$). In item (a) of the next proposition we establish such a monotonicity assuming that the subgroups H and H' are “close” in terms of smallness of the quotient H/H' .

Proposition 3.16. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism and $H' \subseteq H$ two ϕ -inert subgroups of G .*

(a) *If H/H' is finitely generated, then $\widetilde{\text{ent}}(\phi, H') \leq \widetilde{\text{ent}}(\phi, H)$.*

(b) *If H is finitely generated and $G = T(\phi, H)$, then $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi, H)$.*

Proof. (a) Consider the following exact sequence

$$0 \rightarrow \frac{H \cap T_n(\phi, H')}{H'} \rightarrow \frac{T_n(\phi, H')}{H'} \rightarrow \frac{T_n(\phi, H)}{H}.$$

As both H and H' are ϕ -inert, all the groups appearing in the above sequence are finite. This implies that for every $n \in \mathbb{N}_+$, the following inequality holds

$$\log \left| \frac{T_n(\phi, H')}{H'} \right| - \log \left| \frac{H \cap T_n(\phi, H')}{H'} \right| \leq \log \left| \frac{T_n(\phi, H)}{H} \right|.$$

The subgroup $(H/H') \cap (T(\phi, H')/H')$ of the finitely generated group H/H' is finitely generated. Since it is a subgroup of the torsion group $T(\phi, H')/H'$ (see Lemma 2.1(b)), one has $|(H/H') \cap (T(\phi, H')/H')| < \infty$. For every $n \in \mathbb{N}_+$ there is an inclusion $(H \cap T_n(\phi, H'))/H' \subseteq (H/H') \cap (T(\phi, H')/H')$, which implies

$$0 \leq \lim_{n \rightarrow \infty} \frac{\log |(H \cap T_n(\phi, H'))/H'|}{n} \leq \lim_{n \rightarrow \infty} \frac{\log |(H/H') \cap (T(\phi, H')/H')|}{n} = 0.$$

Hence we obtain:

$$\begin{aligned}
\widetilde{\text{ent}}(\phi, H') &= \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, H')/H'|}{n} \\
&= \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, H')/H'|}{n} - \lim_{n \rightarrow \infty} \frac{\log |(H \cap T_n(\phi, H'))/H'|}{n} \\
&= \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, H')/H'| - \log |(H \cap T_n(\phi, H'))/H'|}{n} \\
&\leq \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, H)/H|}{n} = \widetilde{\text{ent}}(\phi, H).
\end{aligned}$$

(b) Given a ϕ -inert subgroup K of G , we have to prove that $\widetilde{\text{ent}}(\phi, K) \leq \widetilde{\text{ent}}(\phi, H)$. First of all, we can suppose that $H \leq K$, as $\widetilde{\text{ent}}(\phi, K) \leq \widetilde{\text{ent}}(\phi, H + K)$ by item (a) (being $(H + K)/K$ finitely generated). By Proposition 2.2, there exists a finitely generated subgroup F of G such that

$$\frac{T_n(\phi, K)}{K} = \frac{K + T_n(\phi, F)}{K},$$

for every $n \in \mathbb{N}_+$. Since $G = T(\phi, H)$ and F is finitely generated, there exists $k \in \mathbb{N}_+$ such that $F \leq T_k(\phi, H)$ and so we obtain that

$$\frac{T_n(\phi, K)}{K} = \frac{K + T_n(\phi, F)}{K} \subseteq \frac{K + T_{n+k}(\phi, H)}{K},$$

for every $n \in \mathbb{N}_+$. Since $(K + T_{n+k}(\phi, H))/K$ is a quotient of $T_{n+k}(\phi, H)/H$ (as we supposed that $H \leq K$), the inequality

$$\left| \frac{T_n(\phi, K)}{K} \right| \leq \left| \frac{T_{n+k}(\phi, H)}{H} \right|$$

holds for every $n \in \mathbb{N}_+$. Applying the logarithm, dividing by n and taking the limit we get $\widetilde{\text{ent}}(\phi, K) \leq \widetilde{\text{ent}}(\phi, T_k(\phi, H))$. To conclude it is enough to apply Lemma 3.11(a). \square

Part (b) of the above proposition is particularly important as it says that, given a ϕ -inert finitely generated subgroup H of G , the quantity $\widetilde{\text{ent}}(\phi, H)$ does not depend on H but only on $T(\phi, H)$, that is, for any other finitely generated ϕ -inert subgroup H' of G with $T(\phi, H) = T(\phi, H')$, one has $\widetilde{\text{ent}}(\phi, H) = \widetilde{\text{ent}}(\phi, H') = \widetilde{\text{ent}}(\phi \upharpoonright_{T(\phi, H)})$.

4 Relationship with the Algebraic Yuzvinski Formula

For $m \in \mathbb{N}_+$, an endomorphism $\phi : \mathbb{Q}^m \rightarrow \mathbb{Q}^m$ is described by an $m \times m$ matrix with coefficients in \mathbb{Q} , which has its monic characteristic polynomial $f_\phi(X) \in \mathbb{Q}[X]$; we say that $f_\phi(X)$ is the characteristic polynomial of ϕ over \mathbb{Q} . Let s be the least positive integer such that $sf_\phi(X) \in \mathbb{Z}[X]$ (i.e., s is the least common multiple of the denominators of the coefficients of $f_\phi(X)$). We call $p_\phi(X) = sf_\phi(X)$ the characteristic polynomial of ϕ over \mathbb{Z} . The roots of $f_\phi(X)$ (which are the same of $p_\phi(X)$) are the eigenvalues $\{\lambda_i : i = 1, \dots, m\}$ of ϕ .

Recall that the quantity

$$\mathfrak{m}(p_\phi(X)) = \log s + \sum_{|\lambda_i| > 1} \log |\lambda_i|$$

is the (logarithmic) *Mahler measure* of $p_\phi(X)$ (see [13], [14] and [12] for more details). The *Mahler measure* of ϕ is $\mathfrak{m}(\phi) = \mathfrak{m}(p_\phi(X))$.

The main result of the recent paper [12] is the following formula concerning the algebraic entropy h of an endomorphism ϕ of \mathbb{Q}^m .

Theorem 4.1 (Algebraic Yuzvinski Formula). *Let m be a positive integer and $\phi : \mathbb{Q}^m \rightarrow \mathbb{Q}^m$ an endomorphism. Then*

$$h(\phi) = \mathfrak{m}(\phi).$$

While the eigenvalues appearing in the Mahler measure have a clear dynamical meaning in terms of “expansiveness” of ϕ , the appearance of $\log s$ is somehow surprising. The main result of this section is the following algebraic interpretation of this mysterious term, generalizing the 1-dimensional case considered in Example 3.4(2).

Theorem 4.2. *Let m be a positive integer, $\phi : \mathbb{Q}^m \rightarrow \mathbb{Q}^m$ an endomorphism and let s be the positive leading coefficient of the characteristic polynomial of ϕ over \mathbb{Z} . Then*

$$\log s = \widetilde{\text{ent}}(\phi).$$

Actually, we prove that $\log s = \widetilde{\text{ent}}(\phi, \mathbb{Z}^m)$, from which Theorem 4.2 follows. In fact, $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi \upharpoonright_{T(\phi, \mathbb{Z}^m)})$ by Corollary 3.15 and $\widetilde{\text{ent}}(\phi \upharpoonright_{T(\phi, \mathbb{Z}^m)}) = \widetilde{\text{ent}}(\phi, \mathbb{Z}^m)$ by Proposition 3.16(b). The proof of the equality $\log s = \widetilde{\text{ent}}(\phi, \mathbb{Z}^m)$, deferred to §4.2, will follow as a consequence of some results from [12].

A consequence of Theorems 4.1 and 4.2 is that the difference between the algebraic entropy h and the intrinsic algebraic entropy of an endomorphism ϕ of a torsion-free Abelian group G of finite rank, depends only on the eigenvalues of the endomorphism, that is

$$h(\phi) - \widetilde{\text{ent}}(\phi) = \sum_{|\lambda_i| > 1} \log |\lambda_i|.$$

4.1 Algebraic entropy for locally compact Abelian groups

For a LCA group G denote by $\mathcal{C}(G)$ the family of all compact neighborhoods of 0 in G and note that $\mathcal{C}(G)$ contains the family $\mathcal{B}(G)$ of all open compact subgroups of G . Let μ be a Haar measure on G .

For a continuous endomorphism $\phi : G \rightarrow G$, the *algebraic entropy of ϕ with respect to $C \in \mathcal{C}(G)$* is

$$H(\phi, C) = \limsup_{n \rightarrow \infty} \frac{\log \mu(T_n(\phi, C))}{n},$$

and the *algebraic entropy of ϕ* is

$$h(\phi) = \sup\{H(\phi, C) : C \in \mathcal{C}(G)\}.$$

Our strategy for the proof of Theorem 4.2 is to find a connection between h and $\widetilde{\text{ent}}$, which holds for continuous endomorphisms of totally disconnected LCA groups, implying in particular that $h(\phi_p) = \widetilde{\text{ent}}(\phi_p)$ for all $p \in \mathbb{P}$ (Theorem 4.3 and Corollary 4.7). Then, we show that $\widetilde{\text{ent}}(\phi, \mathbb{Z}^m) = \sum_{p \in \mathbb{P}} \widetilde{\text{ent}}(\phi_p, \mathbb{J}_p^m)$ (Lemma 4.8).

Theorem 4.3. *Let G be a LCA group and $\phi : G \rightarrow G$ be a continuous endomorphism. Then $\mathcal{B}(G) \subseteq \mathcal{I}_\phi(G)$. Furthermore,*

$$H(\phi, U) = \widetilde{\text{ent}}(\phi, U), \tag{4.1}$$

for all $U \in \mathcal{B}(G)$.

Proof. Let $U \in \mathcal{B}(G)$. Then the subgroup U of the compact subgroup $T_n(\phi, U)$ of G is open, so $T_n(\phi, U)/U$ is finite. Consider now a Haar measure μ on G . Since $T_n(\phi, U)$ is the disjoint union of $|T_n(\phi, U)/U|$ -many cosets of the form $x + U$, with $x \in T_n(\phi, U)$, one has

$$\mu(T_n(\phi, U)) = \mu(U) \cdot |T_n(\phi, U)/U|.$$

Moreover, $0 < \mu(U) < \infty$, as U is open and compact; in particular, $\lim_{n \rightarrow \infty} \frac{\log \mu(U)}{n} = 0$. Therefore,

$$H(\phi, U) = \lim_{n \rightarrow \infty} \frac{\log \mu(U) + \log |T_n(\phi, U)/U|}{n} = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, U)/U|}{n} = \widetilde{\text{ent}}(\phi, U),$$

and this concludes the proof. \square

Remark 4.4. In the sequel G will be a totally disconnected locally compact group G . Recall that for such a group G the family $\mathcal{B}(G)$ is a local base at the neutral element, according to a theorem of van Dantzig [23].

- (a) If the group G is discrete, then the open compact subgroups of G are necessarily finite, so the equality (4.1) in Theorem 4.3 obviously holds true.
- (b) In case the totally disconnected group G is covered by compact subgroups, then by taking the supremum in (4.1), with U running over $\mathcal{B}(G)$, one gets the algebraic entropy $h(\phi)$ on the left hand side. On the right hand side one gets a quantity that is $\leq \widetilde{\text{ent}}(\phi)$. Therefore,

$$h(\phi) \leq \widetilde{\text{ent}}(\phi). \quad (4.2)$$

Furthermore, if G is discrete, then the current hypothesis entails that G is torsion, in which case the opposite inequality in (4.2) is proved in Proposition 3.6.

The next example shows that the inequality in (4.2) can be strict in the non-discrete case.

Example 4.5. Let p be a prime. If $\xi \in \mathbb{J}_p$ is a p -adic integer transcendental over \mathbb{Z} , then the multiplication ϕ_ξ by ξ in \mathbb{Q}_p (equipped with the p -adic topology) has algebraic entropy zero as proved in [7, Example 5.6.5], while its intrinsic algebraic entropy is ∞ . Indeed, ϕ_ξ restricted to the subgroup $\mathbb{Z}[\xi]$ is conjugated to the right Bernoulli shift and so it suffices to apply Lemma 3.9, Lemma 3.10(a) and Example 3.7.

4.2 Proof of Theorem 4.2

The following Proposition 4.6 is an easy consequence of two results of [12], that is, Theorem 3.8 and the Corollary stated in the Introduction. In order to state this result we need to recall some notations.

Let m be a positive integer and $\phi : \mathbb{Q}^m \rightarrow \mathbb{Q}^m$ an endomorphism. For every $p \in \mathbb{P}$, ϕ induces an endomorphism $\phi_p : \mathbb{Q}_p^m \rightarrow \mathbb{Q}_p^m$ just extending the scalars, that is,

$$\phi_p = \phi \otimes_{\mathbb{Q}} \text{id}_{\mathbb{Q}_p}.$$

Proposition 4.6. [12] *Let m be a positive integer, $\phi : \mathbb{Q}^m \rightarrow \mathbb{Q}^m$ an endomorphism and $p_\phi(X) = sX^m + a_1X^{m-1} + \dots + a_m$ the characteristic polynomial over \mathbb{Z} of ϕ . Then*

$$h(\phi_p) = H(\phi_p, \mathbb{J}_p^m) = \log |1/s|_p.$$

Using this proposition we can compute the algebraic entropy h of the various ϕ_p . From this result and from Theorem 4.3 we derive the following consequence.

Corollary 4.7. *Let m be a positive integer, $\phi : \mathbb{Q}^m \rightarrow \mathbb{Q}^m$ an endomorphism and, for a fixed prime p , let $\phi_p : \mathbb{Q}_p^m \rightarrow \mathbb{Q}_p^m$ the induced endomorphism. Then*

$$h(\phi_p) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi_p, \mathbb{J}_p^m)/\mathbb{J}_p^m|}{n} = \widetilde{\text{ent}}(\phi_p, \mathbb{J}_p^m).$$

Proof. By Proposition 4.6, we have that $h(\phi_p) = H(\phi_p, \mathbb{J}_p^m)$. To conclude, apply Theorem 4.3 with $G = \mathbb{Q}_p^m$, $U = \mathbb{J}_p^m$ and $\phi = \phi_p$. \square

The next lemma is the last step for the proof of Theorem 4.2. Remember that \mathbb{J}_p is a valuation domain, furthermore it is flat when considered as a module over \mathbb{Z} . The field of quotients of \mathbb{J}_p is the field \mathbb{Q}_p of p -adic numbers.

Lemma 4.8. *For every $n \in \mathbb{N}_+$, we have that*

$$|T_n(\phi, \mathbb{Z}^m)/\mathbb{Z}^m| = \prod_{p \in \mathbb{P}} |T_n(\phi_p, \mathbb{J}_p^m)/\mathbb{J}_p^m|,$$

where on the right hand side only finitely many factors are > 1 .

Proof. In order to simplify the notation, we set $Q_n = T_n(\phi, \mathbb{Z}^m)/\mathbb{Z}^m$. Consider the exact sequence

$$0 \rightarrow \mathbb{Z}^m \rightarrow T_n(\phi, \mathbb{Z}^m) \rightarrow Q_n \rightarrow 0.$$

Let $p \in \mathbb{P}$; as we said \mathbb{J}_p is a flat \mathbb{Z} -module. In particular the following sequence of \mathbb{J}_p -modules is exact

$$0 \rightarrow \mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{J}_p \rightarrow T_n(\phi, \mathbb{Z}^m) \otimes_{\mathbb{Z}} \mathbb{J}_p \rightarrow Q_n \otimes_{\mathbb{Z}} \mathbb{J}_p \rightarrow 0. \quad (4.3)$$

Since $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{J}_p \cong \mathbb{Q}_p$, we can identify $\mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{J}_p$ and $T_n(\phi, \mathbb{Z}^m) \otimes_{\mathbb{Z}} \mathbb{J}_p$ with \mathbb{J}_p -submodules of \mathbb{Q}_p^m . Fix the canonical base $\{e_1, \dots, e_m\}$ of \mathbb{Q}^m as a \mathbb{Q} -vector space. Then $\{e_1 \otimes 1_{\mathbb{J}_p}, \dots, e_m \otimes 1_{\mathbb{J}_p}\}$ is a base of \mathbb{Q}_p^m as a \mathbb{Q}_p -vector space.

Now, $\mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{J}_p = \mathbb{J}_p^m$ is the \mathbb{J}_p -submodule of \mathbb{Q}_p^m generated by $e_1 \otimes 1_{\mathbb{J}_p}, \dots, e_m \otimes 1_{\mathbb{J}_p}$. Similarly, for every $k \in \mathbb{N}$, $\phi^k(\mathbb{Z}^m)$ is the subgroup of \mathbb{Q}^m generated by $\phi^k(e_1), \dots, \phi^k(e_m)$. This shows that

$$\begin{aligned} (\phi^k \mathbb{Z}^m) \otimes_{\mathbb{Z}} \mathbb{J}_p &= (\phi^k(e_1) \otimes 1_{\mathbb{J}_p})\mathbb{J}_p + \dots + (\phi^k(e_m) \otimes 1_{\mathbb{J}_p})\mathbb{J}_p \\ &= (\phi_p^k(e_1 \otimes 1_{\mathbb{J}_p}))\mathbb{J}_p + \dots + (\phi_p^k(e_m \otimes 1_{\mathbb{J}_p}))\mathbb{J}_p = \phi_p^k(\mathbb{J}_p^m). \end{aligned}$$

In a similar way one can show that $T_n(\phi, \mathbb{Z}^m) \otimes_{\mathbb{Z}} \mathbb{J}_p = T_n(\phi_p, \mathbb{J}_p^m)$ for every $n \in \mathbb{N}_+$.

Since \mathbb{J}_p is q -divisible for every prime $q \neq p$, we have that $Q_n \otimes_{\mathbb{Z}} \mathbb{J}_p$ is isomorphic to the p -component $t_p(Q_n)$ of Q_n . Hence equation (4.3) becomes

$$0 \rightarrow \mathbb{J}_p^m \rightarrow T_n(\phi_p, \mathbb{J}_p^m) \rightarrow t_p(Q_n) \rightarrow 0.$$

It follows that

$$|T_n(\phi, \mathbb{Z}^m)/\mathbb{Z}^m| = |Q_n| = \left| \bigoplus_{p \in \mathbb{P}} t_p(Q_n) \right| = \prod_{p \in \mathbb{P}} |t_p(Q_n)| = \prod_{p \in \mathbb{P}} |T_n(\phi, \mathbb{J}_p^m)/\mathbb{J}_p^m|,$$

as desired. Since Q_n is finite, the family of primes p such that $t_p(Q_n)$ is non-zero is a finite subset of \mathbb{P} . In particular, in the product appearing on the right hand side of the statement, only a finite number of terms are > 1 . \square

In the following remark we give another explication to the last claim in Lemma 4.8.

Remark 4.9. Let $A = (a_{ij})$ be the matrix of ϕ with respect to the canonical base of \mathbb{Q}^m . The coefficients of A are all rational numbers; for every $i, j = 1, \dots, m$ we let $a_{ij} = m_{ij}/d_{ij}$ with $m_{ij}, d_{ij} \in \mathbb{Z}$ and $(m_{ij}, d_{ij}) = 1$. Suppose first that all the coefficients of A are integers, that is, $d_{ij} = 1$ for every i, j . Then the subgroup \mathbb{Z}^m of \mathbb{Q}^m is ϕ -invariant and so $T_n(\phi, \mathbb{Z}^m) = \mathbb{Z}^m$ for every $n \in \mathbb{N}_+$. Hence $|T_n(\phi, \mathbb{Z}^m)/\mathbb{Z}^m| = |\mathbb{Z}^m/\mathbb{Z}^m| = 1$. (In this case we have $\log s = 0$.) Similarly, suppose that $p \in \mathbb{P}$ does

not divide any d_{ij} with $i, j = 1, \dots, m$. Denote by \mathbb{Z}_p the localization of \mathbb{Z} at the prime ideal $p\mathbb{Z}$. It is clear in this situation that \mathbb{Z}_p^m is ϕ -invariant. In particular, $|T_n(\phi, \mathbb{Z}_p^m)/\mathbb{Z}_p^m| = |\mathbb{Z}_p^m/\mathbb{Z}_p^m| = 1$. Since $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{J}_p = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{J}_p$, if \mathbb{Z}_p^m is ϕ -invariant in \mathbb{Q}^m then \mathbb{J}_p^m is ϕ_p -invariant in \mathbb{Q}_p^m . In particular, in the above situation, $|T_n(\phi_p, \mathbb{J}_p^m)/\mathbb{J}_p^m| = |\mathbb{J}_p^m/\mathbb{J}_p^m| = 1$.

Thus we have the following inclusion

$$\{p \in \mathbb{P} : |T_n(\phi_p, \mathbb{J}_p^m)/\mathbb{J}_p^m| > 1\} \subseteq \{p \in \mathbb{P} : p|d_{ij} \text{ for some } i, j = 1, \dots, m\}.$$

We are now in position to demonstrate Theorem 4.2.

Proof of Theorem 4.2. By Lemma 4.8 we get

$$\widetilde{\text{ent}}(\phi, \mathbb{Z}^m) = \sum_{p \in \mathbb{P}} \widetilde{\text{ent}}(\phi_p, \mathbb{J}_p^m). \quad (4.4)$$

Now, Corollary 4.7 and Proposition 4.6 give

$$\widetilde{\text{ent}}(\phi_p, \mathbb{J}_p^m) = h(\phi_p) = \log |1/s|_p.$$

Since $s = \prod_{p \in \mathbb{P}} |1/s|_p$, we can conclude with (4.4) that $\widetilde{\text{ent}}(\phi, \mathbb{Z}^m) = \log s$, as desired. \square

5 The Addition Theorem

We start this section by recalling some notions concerning real-valued invariants of a category of modules. Let R be a ring and $L : \text{Mod}(R) \rightarrow \mathbb{R}^*$ a map; we say that L is an *invariant* if $L(0) = 0$ and $L(M) = L(M')$ whenever $M \cong M'$; and that L is an *additive invariant* if, in addition, $L(M) = L(M') + L(M'')$ for any short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ in $\text{Mod}(R)$.

Clearly, if L is an additive invariant of $\text{Mod}(R)$ and M is an R -module, then, given a finite chain of submodules $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$, we have that $L(M) = \sum_{i=1}^n L(M_i/M_{i-1})$.

An invariant $L : \text{Mod}(R) \rightarrow \mathbb{R}^*$ is *upper continuous* if

$$L(M) = \sup\{L(F) : F \in \mathfrak{F}(M)\},$$

for every $M \in \text{Mod}(R)$, where $\mathfrak{F}(M)$ denotes the family of all the finitely generated submodules of M . If L is both additive and upper continuous, then L is called a *length function*. Length functions have been introduced by Northcott-Reufel [15] and were investigated also by Vámos in [21] and [22].

The following result is contained in [22, Proposition 8].

Lemma 5.1. [22] *Given a ring R and an additive invariant $L : \text{Mod}(R) \rightarrow \mathbb{R}^*$, the following conditions are equivalent:*

- (a) *L is upper continuous;*
- (b) *for every R -module M and every directed system $\{M_i : i \in I\}$ of submodules of M such that $\bigcup_{i \in I} M_i = M$, we have $L(M) = \sup\{L(M_i) : i \in I\}$;*
- (c) *for every R -module M and every totally ordered set $\{M_i : i \in I\}$ of submodules of M such that $\bigcup_{i \in I} M_i = M$, we have $L(M) = \sup\{L(M_i) : i \in I\}$.*

The meaning of Lemma 5.1 is that any additive invariant which is continuous on direct limits is completely determined by the values it assumes on finitely generated modules. This becomes even more convenient over Noetherian rings (so also for Abelian groups), where the finitely generated modules form a Serre subclass of $\text{Mod}(R)$. In the following proposition we see that an upper continuous invariant is a length function if and only if it is additive on finitely generated modules. This fact will be used in demonstrating the Addition Theorem.

Proposition 5.2. *Let R be a Noetherian ring and $L : \text{Mod}(R) \rightarrow \mathbb{R}^*$ be an upper continuous invariant. If $L(F_2) = L(F_1) + L(F_3)$ for every short exact sequence $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$ of finitely generated R -modules, then L is additive, hence a length function.*

Proof. Since L is upper continuous by hypothesis, it remains only to check additivity. To this end, let N and M be R -modules such that $N \subseteq M$, and choose $F \in \mathfrak{F}(M)$; so $F \cap N \in \mathfrak{F}(N)$ since R is Noetherian, and $(F + N)/N \in \mathfrak{F}(M/N)$. By the additivity of L on finitely generated modules, we obtain $L(F) = L(F \cap N) + L((F + N)/N)$ which proves that $L(M) \leq L(N) + L(M/N)$ by the arbitrariness of F .

On the other hand, let $F_1 \in \mathfrak{F}(N)$ and $\overline{F}_2 \in \mathfrak{F}(M/N)$. Choose a submodule $F_2 \in \mathfrak{F}(M)$ such that $(F_2 + N)/N = \overline{F}_2$ and set $F = F_1 + F_2$. Then

$$L(F) = L(F \cap N) + L((F + N)/N) \geq L(F_1) + L(\overline{F}_2),$$

since $\overline{F}_2 = (F + N)/N$ (being F_1 contained in N) and $F_1 \subseteq F \cap N$. This gives $L(M) \geq L(N) + L(M/N)$, concluding the proof. \square

Usually a mathematical notion of entropy (no matter if topological, algebraic, measure-theoretic) is defined as a map from the set of endomorphisms of a fixed object in a category \mathfrak{C} to \mathbb{R}^* . This point of view can be made categorical as follows (see [6] and [20]).

Given a category \mathfrak{C} , let $\text{Flow}(\mathfrak{C})$ denote the category of *flows* of \mathfrak{C} , whose objects are the pairs (G, ϕ) where G is an object of \mathfrak{C} and $\phi : G \rightarrow G$ is an endomorphism. A morphism $(G_1, \phi) \rightarrow (G_2, \psi)$ in $\text{Flow}(\mathfrak{C})$ is a commutative square

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi} & G_1 \\ \alpha \downarrow & & \downarrow \alpha \\ G_2 & \xrightarrow{\psi} & G_2 \end{array}$$

where $\alpha : G_1 \rightarrow G_2$ is a morphism in \mathfrak{C} . Abusing notation, we shall denote by α also the morphism $(G_1, \phi_1) \rightarrow (G_2, \phi_2)$ in $\text{Flow}(\mathfrak{C})$.

When the category \mathfrak{C} is the category $\text{Mod}(R)$, there is a nice description of the category of flows; in fact, the category $\text{Flow}(\text{Mod}(R))$ is canonically isomorphic to $\text{Mod}(R[X])$. The canonical isomorphism is given by $(G, \phi) \mapsto G_\phi \in \text{Mod}(R[X])$, where G_ϕ as an R -module is just G and X acts on G via ϕ , that is

$$(r_0 + r_1X + \cdots + r_nX^n) \cdot g = r_0g + r_1\phi(g) + \cdots + r_n\phi^n(g)$$

for all $g \in G$ and $(r_0 + r_1X + \cdots + r_nX^n) \in R[X]$. Also a morphism in $\text{Flow}(\text{Mod}(R))$ is easily seen to commute with the action of X and thus it becomes an $R[X]$ -module homomorphism.

In what follows we will identify the categories $\text{Flow}(\text{Mod}(R))$ and $\text{Mod}(R[X])$, denoting by G_ϕ a generic object.

Let G_ϕ be an $R[X]$ -module. An $R[X]$ -submodule of G_ϕ is nothing else but a ϕ -invariant submodule of G . Similarly, a quotient of G_ϕ is just a quotient of G such that the kernel of the natural projection

is ϕ -invariant. Recall that a segment of G_ϕ is any submodule of a quotient of G_ϕ . We shall sometimes abuse notation, denoting by H_ϕ a segment of G_ϕ with X acting as the endomorphism induced on H by ϕ .

Notice that it is possible to consider the intrinsic algebraic entropy as an invariant of $\text{Mod}(\mathbb{Z}[X])$. Indeed, we let

$$\widetilde{\text{ent}} : \text{Mod}(\mathbb{Z}[X]) \rightarrow \mathbb{R}^*, \quad G_\phi \mapsto \widetilde{\text{ent}}(\phi).$$

By Lemma 3.9 $\widetilde{\text{ent}}$ is an invariant and by Lemma 3.14 $\widetilde{\text{ent}}$ is also upper continuous.

The remaining part of this section is devoted to prove the Addition Theorem for $\widetilde{\text{ent}}$, which says that, given a $\mathbb{Z}[X]$ -module G_ϕ and a $\mathbb{Z}[X]$ -submodule H_ϕ of G_ϕ , then $\widetilde{\text{ent}}(G_\phi) = \widetilde{\text{ent}}(H_\phi) + \widetilde{\text{ent}}((G/H)_\phi)$.

The next two results give the proof of the Addition Theorem in the very particular case of a divisible torsion-free Abelian group of finite rank with a pure ϕ -invariant subgroup. The proof of the next lemma is analogous to that of [6, Proposition 4.2] in the case of h .

Lemma 5.3. *Let m be a positive integer and $\phi : \mathbb{Q}^m \rightarrow \mathbb{Q}^m$ an endomorphism. Suppose that $K \cong \mathbb{Q}^k$, for some $k \leq m$, is a ϕ -invariant \mathbb{Q} -vector subspace of \mathbb{Q}^m . Then*

$$\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi|_K) + \widetilde{\text{ent}}(\bar{\phi}),$$

where $\bar{\phi} : \mathbb{Q}^m/K \rightarrow \mathbb{Q}^m/K$ is the endomorphism induced by ϕ .

Proof. Choose a basis $\mathcal{B} = \{v_1, \dots, v_k, v_{k+1}, \dots, v_m\}$ of \mathbb{Q}^m such that $\mathcal{B}_K = \{v_1, \dots, v_k\}$ is a basis of K ; the matrix of ϕ with respect to \mathcal{B} has the following block-wise form:

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix},$$

where A_1 is the matrix of $\phi|_K$ with respect to \mathcal{B}_K . Let $\pi : \mathbb{Q}^m \rightarrow \mathbb{Q}^m/K$ be the canonical projection and let $\bar{\mathcal{B}} = \{\pi(v_{k+1}), \dots, \pi(v_m)\}$, which is a basis of \mathbb{Q}^m/K . Then A_2 is the matrix of $\bar{\phi}$ with respect to $\bar{\mathcal{B}}$. Let $\alpha_1, \dots, \alpha_k$ be the eigenvalues of A_1 and let $\alpha_{k+1}, \dots, \alpha_m$ be the eigenvalues of A_2 .

Let $f_\phi(X), f_1(X), f_2(X) \in \mathbb{Q}[X]$ be the characteristic polynomials of A, A_1, A_2 respectively. Then $f_\phi(X) = f_1(X) \cdot f_2(X)$. Let s_1 and s_2 be the least common multiples of the denominators of the coefficients of f_1 and f_2 respectively. This means that $p_1(X) = s_1 f_1(X)$ and $p_2(X) = s_2 f_2(X) \in \mathbb{Z}[X]$ are primitive. By Gauss Lemma, $p_\phi(X) = p_1(X) \cdot p_2(X)$ is primitive and so, setting $s = s_1 s_2$, the polynomial $p_\phi(X) = s f_\phi(X) \in \mathbb{Z}[X]$ is primitive. Now Theorem 4.2 applied to $\phi, \phi|_K$ and $\bar{\phi}$ gives $\widetilde{\text{ent}}(\phi) = \log s = \log s_1 + \log s_2 = \widetilde{\text{ent}}(\phi|_K) + \widetilde{\text{ent}}(\bar{\phi})$. \square

Proposition 5.4. *Let G be a torsion-free Abelian group of finite rank and $\phi : G \rightarrow G$ an endomorphism. If H is a ϕ -invariant pure subgroup of G then*

$$\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi|_H) + \widetilde{\text{ent}}(\bar{\phi}),$$

where $\bar{\phi} : G/H \rightarrow G/H$ is the morphism induced by ϕ .

Proof. As H is pure in G we have that G/H is torsion-free. Furthermore, the following exact sequence

$$0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0$$

is sent to the following short exact sequence tensorizing by \mathbb{Q} :

$$0 \rightarrow D(H) \rightarrow D(G) \rightarrow D(G/H) \rightarrow 0.$$

Now, to conclude the proof it is enough to use the fact that the intrinsic algebraic entropy is preserved by taking divisible hulls of torsion-free Abelian groups by Corollary 3.15, and that it is additive on sequences of rational vector spaces by Lemma 5.3. \square

Our next goal is to prove the Addition Theorem in the particular case when G is a finite-rank torsion-free Abelian group that is the trajectory of a finitely generated subgroup, and H is a ϕ -invariant subgroup of G of maximal rank.

Proposition 5.5. *Let G be a torsion-free Abelian group of finite rank $k \in \mathbb{N}_+$, $\phi : G \rightarrow G$ an endomorphism and suppose there exists a finitely generated subgroup F of G such that $G = T(\phi, F)$. Let H be a ϕ -invariant subgroup of G such that G/H is torsion. Then:*

- (a) G/H is finite;
- (b) $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi \upharpoonright_H)$ and $\widetilde{\text{ent}}(\bar{\phi}) = 0$, where $\bar{\phi}$ is the endomorphism induced by ϕ on G/H .

Proof. There exist finitely many prime numbers $\{p_1, \dots, p_h\}$ such that, identifying G with a subgroup of its divisible hull $D(G) = \mathbb{Q}^k$, we have $G \subseteq \mathbb{Z}[1/p_1, \dots, 1/p_h]^k$, where $\mathbb{Z}[1/p_1, \dots, 1/p_h] = R$ is the subring of \mathbb{Q} generated by the fractions $1/p_i$ with $1 \leq i \leq h$. Indeed, since F is finitely generated, we can suppose $F \cong \mathbb{Z}^m \subseteq \mathbb{Z}^k$ for some $m \leq k$. Let $\{p_1, \dots, p_h\}$ be the family of primes dividing the coefficients of the rational matrix associated to the extension ϕ of ϕ to $D(G)$. Then clearly $\phi^n(F) \subseteq R^k$ for every $n \in \mathbb{N}$, hence $G = T(\phi, F) \subseteq R^k$.

(a) As $(G/H)_\phi$ is a finitely generated $\mathbb{Z}[X]$ -module, any positive integer annihilating its generators annihilates G/H , which is therefore bounded. A bounded Abelian group with finite socle is finite, so it is enough to prove that G/H has finite socle. Since the class of torsion Abelian groups with finite socle is closed under taking subgroups, quotients and finite direct sums, recalling that G/H is contained into R^k/H , and noting that R^k/H is a quotient of $R/m_1\mathbb{Z} \oplus \dots \oplus R/m_k\mathbb{Z}$ for suitable non-zero integers m_1, \dots, m_k , the conclusion comes from the fact that the socle of $R/n\mathbb{Z}$ is finite for all $n \in \mathbb{Z}$, because $R/n\mathbb{Z}$ is a finite direct sum of cocyclic groups.

(b) As G and H have the same rank, $D(G) = D(H)$ and so it follows by Corollary 3.15 that $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi \upharpoonright_H)$. Furthermore, G/H is finite by item (a) and so $\widetilde{\text{ent}}(\bar{\phi}) = 0$. \square

We pass now to examine Abelian groups which fail to be torsion-free, but which still are trajectories of finitely generated subgroups. Proposition 3.16(b) shows that, if the Abelian group G is the trajectory of a finitely generated ϕ -inert subgroup H , then $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi, H)$. On the other hand, it is possible to find examples where G is the trajectory of a finitely generated subgroup, but it is not possible to find a finite set of elements generating a ϕ -inert subgroup (take for example the right Bernoulli shift on the group $\mathbb{Z}^{(\mathbb{N})}$). We will prove in the following proposition that this inconvenience may occur only if the torsion-free rank of G is infinite.

Proposition 5.6. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism and suppose that there exists a finitely generated subgroup F of G such that $G = T(\phi, F)$. Then the following conditions are equivalent:*

- (a) $\widetilde{\text{ent}}(\phi) < \infty$;
- (b) $\text{rk}(G) < \infty$;
- (c) there exists $n \in \mathbb{N}_+$ such that $T_n(\phi, F)$ is ϕ -inert.

Furthermore, if the above equivalent conditions are not verified, then G has a ϕ -inert subgroup isomorphic to $\mathbb{Z}^{(\mathbb{N})}$ on which the restriction of ϕ acts as the right Bernoulli shift $\beta_{\mathbb{Z}}$.

Proof. (a) \Rightarrow (b) Suppose $\text{rk}(G) = \infty$. Let x_1, \dots, x_k be a set of generators of F , in particular, $G = T(\phi, x_1\mathbb{Z}) + \dots + T(\phi, x_k\mathbb{Z})$ and so $\text{rk}(G) \leq \sum_{i=1}^k \text{rk}(T(\phi, x_i\mathbb{Z}))$. In particular, $\text{rk}(T(\phi, x_i\mathbb{Z})) = \infty$ for some i . But the condition that $\text{rk}(T(\phi, x_i\mathbb{Z})) = \infty$ is equivalent to the fact that $\phi^n(x_i)\mathbb{Z} \cong \mathbb{Z}$ and

$T_n(\phi, x_i) \cong \mathbb{Z}^n$, for every $n \in \mathbb{N}_+$. This means exactly that $T(\phi, x_i\mathbb{Z}) \cong \mathbb{Z}^{(\mathbb{N})}$ and ϕ acts on this group as a right Bernoulli shift. From Example 3.7 we deduce that $\text{ent}(\phi) = \infty$.

(b) \Rightarrow (c) There exists a finitely generated subgroup F' of G such that $F' \cong \mathbb{Z}^k$ and G/F' is torsion. Since $G = T(\phi, F)$ and F' is finitely generated, there exists $n \in \mathbb{N}_+$ such that $F' \subseteq T_n(\phi, F)$. Now $(T_n(\phi, F) + \phi(T_n(\phi, F)))/T_n(\phi, F)$ is torsion, because it is contained in a quotient of G/F' , and also finitely generated, hence it is finite, so $T_n(\phi, F)$ is ϕ -inert as desired.

(c) \Rightarrow (a) Since $T_n(\phi, F)$ is also finitely generated, by Proposition 3.16(b) one can see that $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi, T_n(\phi, F))$, therefore,

$$\widetilde{\text{ent}}(\phi) \leq \log |T_{n+1}(\phi, F)/T_n(\phi, F)| < \infty,$$

and this concludes the proof. \square

The following technical lemma is needed in the proof of Proposition 5.8

Lemma 5.7. *Let G be an Abelian group of finite rank, $\phi : G \rightarrow G$ an endomorphism and suppose there exists a finitely generated subgroup F of G such that $G = T(\phi, F)$. Then $t(G)$ is bounded, hence $G = t(G) \oplus K$, where K is torsion-free. Hence ϕ can be represented in matrix form as follows $(\phi_t \in \text{End}(t(G)), \phi^* \in \text{Hom}(K, t(G)), \bar{\phi} \in \text{End}(K))$:*

$$\begin{pmatrix} \phi_t & \phi^* \\ 0 & \bar{\phi} \end{pmatrix} : t(G) \oplus K \rightarrow t(G) \oplus K \quad \begin{pmatrix} t \\ f \end{pmatrix} \mapsto \begin{pmatrix} \phi_t(t) + \phi^*(f) \\ \bar{\phi}(f) \end{pmatrix}.$$

Furthermore, there exist subgroups H_1 of $t(G)$ and H_2 of K such that:

- (i) H_1 is finite, it contains $\phi^*(T(\bar{\phi}, H_2))$ and $T(\phi_t, H_1) = t(G)$;
- (ii) H_2 is finitely generated, $\bar{\phi}$ -inert and $T(\bar{\phi}, H_2) = K$;
- (iii) $H = H_1 \oplus H_2$ is finitely generated, ϕ -inert and $T(\phi, H) = G$;
- (iv) $\widetilde{\text{ent}}(\phi, H) = \widetilde{\text{ent}}(\phi_t, H_1) + \widetilde{\text{ent}}(\bar{\phi}, H_2)$.

Proof. By hypothesis G_ϕ is a finitely generated $\mathbb{Z}[X]$ -module. As $\mathbb{Z}[X]$ is a Noetherian ring, the $\mathbb{Z}[X]$ -submodule $T = t(G)_\phi$ is again finitely generated, so $T = T(\phi, F_1)$ for a finite subgroup F_1 . A positive integer annihilating F_1 annihilates the whole T , which is therefore bounded, hence a summand of G .

Thus, we can adopt the notation of the statement and let $\pi : G \rightarrow K$ be the canonical projection. We have that $K = T(\bar{\phi}, \pi F)$ satisfies the hypothesis of Proposition 5.6 and so, there exists $n \geq 1$ such that $H_2 = T_n(\bar{\phi}, \pi F)$ is a $\bar{\phi}$ -inert subgroup of K . We let also $H_1 = F_1 + \phi^*(T(\bar{\phi}, H_2))$.

(i) As $\phi^*(T(\bar{\phi}, H_2))$ is bounded, there exists $m \geq 1$ such that $m\phi^*(T(\bar{\phi}, H_2)) = 0$. In particular, $mT(\bar{\phi}, H_2) \subseteq \ker(\phi^*) \cap T(\bar{\phi}, H_2)$ and so $\phi^*(T(\bar{\phi}, H_2))$ is a quotient of $T(\bar{\phi}, H_2)/mT(\bar{\phi}, H_2)$, which is a finite group, since $T(\bar{\phi}, H_2)$ has finite rank. Hence H_1 is the sum of two finite groups, thus it is finitely generated and ϕ_t -inert. Furthermore, the inclusions $T = T(\phi_t, F_1) \subseteq T(\phi_t, H_1) \subseteq T$ show that $T(\phi_t, H_1) = T$.

(ii) easily follows by construction.

(iii) The subgroup H is finitely generated as it is the sum of two finitely generated subgroups. Since H_2 is ϕ -inert by part (ii), H is ϕ -inert by Lemma 2.6. Finally, the inclusion $T(\phi_t, H_1) \oplus T(\bar{\phi}, H_2) \subseteq T(\phi, H)$ shows that $G = t(G) \oplus K = T(\phi, H)$.

(iv) We want now to compute the n -trajectories of H . It is useful here to use the matrix notation, with the obvious meaning of symbols. Indeed, we have:

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad \phi(H) \subseteq \begin{pmatrix} \phi_t(H_1) + \phi^*(H_2) \\ \bar{\phi}(H_2) \end{pmatrix}, \quad \phi^2(H) \subseteq \begin{pmatrix} \phi_t^2(H_1) + \phi_t\phi^*(H_2) + \phi^*\bar{\phi}(H_2) \\ \bar{\phi}^2(H_2) \end{pmatrix}, \dots$$

Going on this way, one obtains the inclusion

$$\phi^n(H) \subseteq \begin{pmatrix} \phi_t^n H_1 + \sum \{ \phi_t^i \phi^* \bar{\phi}^j(H_2) : i+j = n-1, 0 \leq i, j \leq n-1 \} \\ \bar{\phi}^n(H_2) \end{pmatrix}$$

It is clear that $\sum \phi_t^i \phi^* \bar{\phi}^j H_2 \subseteq \sum \phi_t^i \phi^*(T(\bar{\phi}, H_2))$. Now recall that by construction, $\phi^*(T(\bar{\phi}, H_2)) \leq H_1$ and so we can improve the above inclusion as follows

$$\phi^n(H) \subseteq \begin{pmatrix} \phi_t^n(H_1) + \sum_{i=0}^{n-1} \phi_t^i(H_1) \\ \bar{\phi}^n(H_2) \end{pmatrix}$$

An easy computation now gives

$$T_n(\phi, H) \subseteq \begin{pmatrix} T_n(\phi_t, H_1) \\ T_n(\bar{\phi}, H_2) \end{pmatrix}.$$

Hence we obtain that $T_n(\phi, H)/H \subseteq T_n(\phi_t, H_1)/H_1 \oplus T_n(\bar{\phi}, H_2)/H_2$. This shows that

$$\widetilde{\text{ent}}(\phi, H) \leq \widetilde{\text{ent}}(\phi_t, H_1) + \widetilde{\text{ent}}(\bar{\phi}, H_2).$$

Since the other inequality holds by Lemma 3.10(b), we get the wanted equality. \square

The next result proves the Addition Theorem with respect to the torsion subgroup.

Proposition 5.8. *Let G be an Abelian group, $\phi : G \rightarrow G$ an endomorphism and $\bar{\phi} : G/t(G) \rightarrow G/t(G)$ the endomorphism induced by ϕ . Suppose there exists a finitely generated subgroup F of G such that $G = T(\phi, F)$. Then $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi \upharpoonright_{t(G)}) + \widetilde{\text{ent}}(\bar{\phi})$.*

Proof. Suppose that $\text{rk}(G)$ is infinite. By Proposition 5.6 $\widetilde{\text{ent}}(\phi) = \infty$ and $G/t(G)$ has a ϕ -invariant subgroup isomorphic to $\mathbb{Z}^{(\mathbb{N})}$ on which $\bar{\phi}$ acts as a right Bernoulli shift, so clearly $\widetilde{\text{ent}}(\phi) = \infty = \widetilde{\text{ent}}(\bar{\phi})$. So assume that $\text{rk}(G)$ is finite. By Lemma 5.7 $t(G)$ splits in G and so $G = t(G) \oplus K$, where K is torsion-free. Again by Lemma 5.7 there exist a finite subgroup H_1 of $t(G)$ and a finitely generated subgroup H_2 of K such that $T(\phi \upharpoonright_{t(G)}, H_1) = t(G)$ and $T(\bar{\phi}, H_2) = K$; moreover $H = H_1 \oplus H_2$ is finitely generated, ϕ -inert, $T(\phi, H) = G$ and $\widetilde{\text{ent}}(\phi, H) = \widetilde{\text{ent}}(\phi_t, H_1) + \widetilde{\text{ent}}(\bar{\phi}, H_2)$. In order to conclude, notice that $\widetilde{\text{ent}}(\phi, H) = \widetilde{\text{ent}}(\phi)$, $\widetilde{\text{ent}}(\phi \upharpoonright_{t(G)}, H_1) = \widetilde{\text{ent}}(\phi_t)$ and $\widetilde{\text{ent}}(\bar{\phi}, H_2) = \widetilde{\text{ent}}(\bar{\phi})$ by Proposition 3.16(b). \square

The preceding results allow us to prove the Addition Theorem for the intrinsic algebraic entropy in case the Abelian group we are dealing with is the trajectory of a finitely generated subgroup. The general case will follow using Proposition 5.2.

Theorem 5.9. *Let G be an Abelian group, ϕ an endomorphism and suppose that there exists a finitely generated subgroup F of G such that $G = T(\phi, F)$. Then, for every ϕ -invariant subgroup H of G we have that*

$$\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi \upharpoonright_H) + \widetilde{\text{ent}}(\bar{\phi}),$$

where $\bar{\phi}$ denotes the endomorphism induced by ϕ on the quotient G/H .

Proof. Obviously, $\text{rk}(G) = \infty$ if and only if $\text{rk}(H) = \infty$ or $\text{rk}(G/H) = \infty$, as $\text{rk}(G) = \text{rk}(H) + \text{rk}(G/H)$. By Proposition 5.6, this means that $\widetilde{\text{ent}}(\phi) = \infty$ if and only if $\widetilde{\text{ent}}(\phi \upharpoonright_H) = \infty$ or $\widetilde{\text{ent}}(\bar{\phi}) = \infty$. This allows us to suppose, as we will do from now on, that G has finite rank. In order to simplify the notation, we will adopt the following conventions. All the groups that will appear in this proof will be of

the form K_1/K_2 , with $K_2 \leq K_1$ ϕ -invariant subgroups of G , so that ϕ induces an endomorphism ϕ_{K_1/K_2} of K_1/K_2 . We will denote $\widetilde{\text{ent}}(\phi_{K_1/K_2})$ simply by $\widetilde{\text{ent}}(K_1/K_2)$. Furthermore, we let $K = t(G) + H$.

Let us list some short exact sequences of $\mathbb{Z}[X]$ -modules on which $\widetilde{\text{ent}}$ is additive (recall that we are assuming G of finite rank).

- (1) From $0 \rightarrow t(G) \rightarrow G \rightarrow G/t(G) \rightarrow 0$ we have that $\widetilde{\text{ent}}(G) = \widetilde{\text{ent}}(t(G)) + \widetilde{\text{ent}}(G/t(G))$ by Proposition 5.8.
- (2) From $0 \rightarrow t(H) \rightarrow H \rightarrow K/t(G) \rightarrow 0$ we have that $\widetilde{\text{ent}}(H) = \widetilde{\text{ent}}(t(H)) + \widetilde{\text{ent}}(K/t(G))$ by Proposition 5.8.
- (3) From $0 \rightarrow t(H) \rightarrow t(G) \rightarrow t(G)/t(H) \rightarrow 0$ we have that $\widetilde{\text{ent}}(t(G)) = \widetilde{\text{ent}}(t(H)) + \widetilde{\text{ent}}(t(G)/t(H))$ by the Addition Theorem for ent (see [9]) and the fact that $\text{ent} = \widetilde{\text{ent}}$ on torsion Abelian groups.
- (4) Considering the short exact sequence in (1.6) (replacing there G by $G/t(G)$ and L by $K/t(G)$), and noting that $(G/t(G))/(K/t(G)) \cong G/K$, we obtain the short exact sequence:

$$0 \rightarrow (K/t(G))^* \rightarrow G/t(G) \rightarrow \frac{G/K}{t(G/K)} \rightarrow 0.$$

So we have that $\widetilde{\text{ent}}(G/t(G)) = \widetilde{\text{ent}}((K/t(G))^*) + \widetilde{\text{ent}}((G/K)/t(G/K))$ by Corollary 5.4.

- (5) From $0 \rightarrow K/t(G) \rightarrow (K/t(G))^* \rightarrow (K/t(G))^*/(K/t(G)) \rightarrow 0$ we have that $\widetilde{\text{ent}}(K/t(G)) = \widetilde{\text{ent}}((K/t(G))^*)$, by Proposition 5.5.
- (6) It is not difficult to verify that $(G/K)/t(G/K) \cong (G/H)/t(G/H)$ (to see this just notice that the two canonical surjections $G \rightarrow (G/K)/t(G/K)$ and $G \rightarrow (G/H)/t(G/H)$ have the same kernel), and so $\widetilde{\text{ent}}((G/K)/t(G/K)) = \widetilde{\text{ent}}((G/H)/t(G/H))$.
- (7) From $0 \rightarrow t(G/H) \rightarrow G/H \rightarrow (G/H)/t(G/H) \rightarrow 0$ we have, again by Proposition 5.8, that $\widetilde{\text{ent}}(G/H) = \widetilde{\text{ent}}(t(G/H)) + \widetilde{\text{ent}}((G/H)/t(G/H))$.
- (8) The quotient K/H is contained in $t(G/H)$ and it is the kernel of the projection $t(G/H) \rightarrow t(G/K)$. Moreover, $K/H \cong t(G)/t(H)$. So we consider the short exact sequence of torsion Abelian groups

$$0 \rightarrow t(G)/t(H) \rightarrow t(G/H) \rightarrow t(G/K) \rightarrow 0.$$

Since

$$t(G/K) \cong t\left(\frac{G/t(G)}{K/t(G)}\right) = \frac{(K/t(G))^*}{K/t(G)},$$

and the latter quotient is finite by Proposition 5.5, we can conclude that $t(G/K)$ is finite. So $\widetilde{\text{ent}}(t(G/K)) = 0$. By the Addition Theorem for ent (see [9]) and the fact that $\widetilde{\text{ent}} = \text{ent}$ on torsion Abelian groups, we have $\widetilde{\text{ent}}(t(G/H)) = \widetilde{\text{ent}}(t(G)/t(H))$.

Now we can conclude the proof with the following series of equalities:

$$\begin{aligned} \widetilde{\text{ent}}(G) &= \widetilde{\text{ent}}(t(G)) + \widetilde{\text{ent}}(G/t(G)) && \text{by (1)} \\ &= \widetilde{\text{ent}}(t(H)) + \widetilde{\text{ent}}(t(G)/t(H)) + \widetilde{\text{ent}}((K/t(G))^*) + \widetilde{\text{ent}}((G/K)/t(G/K)) && \text{by (3) and (4)} \\ &= \widetilde{\text{ent}}(t(H)) + \widetilde{\text{ent}}(t(G)/t(H)) + \widetilde{\text{ent}}(K/t(G)) + \widetilde{\text{ent}}((G/H)/t(G/H)) && \text{by (5) and (6)} \\ &= \widetilde{\text{ent}}(H) + \widetilde{\text{ent}}(t(G)/t(H)) + \widetilde{\text{ent}}(G/H) - \widetilde{\text{ent}}(t(G/H)) && \text{by (2) and (7)} \\ &= \widetilde{\text{ent}}(H) + \widetilde{\text{ent}}(G/H) && \text{by (8).} \end{aligned}$$

This concludes the proof. \square

Corollary 5.10 (Addition Theorem). *The intrinsic algebraic entropy $\widetilde{\text{ent}} : \text{Mod}(\mathbb{Z}[X]) \rightarrow \mathbb{R}^*$ is an additive invariant, hence it is a length function.*

Proof. In Lemma 3.14 we proved that $\widetilde{\text{ent}}$ is an upper continuous invariant, so it is a length function provided that it is additive. The above Theorem 5.9 shows that $\widetilde{\text{ent}}$ is additive on the class of finitely generated $\mathbb{Z}[X]$ -modules. Since $\mathbb{Z}[X]$ is Noetherian, Proposition 5.2 applies to show that $\widetilde{\text{ent}}$ is additive on the whole category $\text{Mod}(\mathbb{Z}[X])$. \square

We can easily derive from the Addition Theorem a characterization of the endomorphisms having intrinsic algebraic entropy zero.

Corollary 5.11. *Let G be an Abelian group and $\phi : G \rightarrow G$ an endomorphism. Then $\widetilde{\text{ent}}(\phi) = 0$ if and only if the following conditions are satisfied:*

(a) $\text{ent}(\phi \upharpoonright_{t(G)}) = 0$;

(b) G is the union of a chain of pure ϕ -invariant subgroups

$$t(G) = G_0 < G_1 < G_2 < \dots < G_\sigma < \dots < \bigcup_{\sigma < \lambda} G_\sigma = G$$

such that $\text{rk}(G_{\sigma+1}/G_\sigma)$ is finite for all σ and, denoting by $\phi_{\sigma+1} : G_{\sigma+1}/G_\sigma \rightarrow G_{\sigma+1}/G_\sigma$ the endomorphism induced by ϕ , $G_{\sigma+1}/G_\sigma$ is the pure closure in G/G_σ of a cyclic $\phi_{\sigma+1}$ -trajectory for all $\sigma < \lambda$;

(c) the unique extension $\widetilde{\phi}_{\sigma+1}$ of $\phi_{\sigma+1}$ to the divisible hull of $G_{\sigma+1}/G_\sigma$ has monic characteristic polynomial over \mathbb{Z} .

Proof. First assume that (a), (b) and (c) are true. We prove that $\widetilde{\text{ent}}(\phi \upharpoonright_{G_\sigma}) = 0$ for each ordinal σ , by transfinite induction on σ ; the fact that $\widetilde{\text{ent}}(\phi) = 0$ then follows by Lemma 5.1. Item (a) provides the initial step for $\sigma = 0$. Assume $\widetilde{\text{ent}}(\phi \upharpoonright_{G_\sigma}) = 0$ and consider the exact sequence:

$$0 \rightarrow G_\sigma \rightarrow G_{\sigma+1} \rightarrow G_{\sigma+1}/G_\sigma \rightarrow 0.$$

The endomorphism $\phi_{\sigma+1}$ of $G_{\sigma+1}/G_\sigma$ induced by $\phi \upharpoonright_{G_{\sigma+1}}$ has zero intrinsic algebraic entropy by (c), Corollary 3.15 and Theorem 4.2; then the Addition Theorem ensures that $\widetilde{\text{ent}}(\phi \upharpoonright_{G_{\sigma+1}}) = 0$. Assume now that σ is a limit ordinal, and that $\widetilde{\text{ent}}(\phi \upharpoonright_{G_\rho}) = 0$ for all ordinals $\rho < \sigma$. Then Lemma 5.1 ensures that also $\widetilde{\text{ent}}(\phi \upharpoonright_{G_\sigma}) = 0$, so we are done.

Conversely, let us assume that $\widetilde{\text{ent}}(\phi) = 0$. Then (a) follows from Lemma 3.10(a). The construction of the chain in (b) is made by transfinite induction. If G_σ is already constructed, let $\psi_\sigma : G/G_\sigma \rightarrow G/G_\sigma$ the map induced by ϕ . Choose $x \in G \setminus G_\sigma$ and consider the ψ_σ -trajectory $T(\psi_\sigma, x + G_\sigma) = T_\sigma$ in G/G_σ ; let $G_{\sigma+1}/G_\sigma = T_\sigma^*$ be its pure closure in G/G_σ . Finally, let $\phi_{\sigma+1} : G_{\sigma+1}/G_\sigma \rightarrow G_{\sigma+1}/G_\sigma$ be the restriction of ψ_σ to $G_{\sigma+1}/G_\sigma$. If σ is a limit ordinal, set $G_\sigma = \bigcup_{\rho < \sigma} G_\rho$. Then $\widetilde{\text{ent}}(\phi) = 0$ implies $\widetilde{\text{ent}}(\psi_{\sigma+1}) = 0$ for all σ , and consequently also that $\widetilde{\text{ent}}(\phi_{\sigma+1}) = 0$ for all σ ; this last equality has the consequence that $\text{rk}(G_{\sigma+1}/G_\sigma)$ is finite, because $T_\sigma = T(\psi_\sigma, x + G_\sigma)$ of infinite rank would imply that $\widetilde{\text{ent}}(\phi_{\sigma+1}) = \infty$, by Example 3.7. Now item (c) follows from Corollary 3.15 and Theorem 4.2. \square

As done in the case of the algebraic entropy in [9], it is possible to define the global intrinsic algebraic entropy of an Abelian group G as

$$\widetilde{\text{ent}}(G) = \sup\{\widetilde{\text{ent}}(\phi) : \phi : G \rightarrow G \text{ endomorphism}\}.$$

The following is an example of a torsion-free Abelian group with $\text{rk}(G) = 2^{\aleph_0}$ and $\widetilde{\text{ent}}(G) = 0$.

Example 5.12. It is well known that there exist 2^{\aleph_0} subgroups R_σ , with $\sigma < 2^{\aleph_0}$, of the group \mathbb{Q} of the rational numbers such that $\text{Hom}_{\mathbb{Z}}(R_\sigma, R_\tau) = 0$ if $\sigma \neq \tau$ and $\text{End}(R_\sigma) = \mathbb{Z}$ for all σ (for instance, take 2^{\aleph_0} infinite subsets σ of \mathbb{P} with finite mutual intersection, and for each σ let $R_\sigma = \langle 1/p : p \in \sigma \rangle$).

Let $G = \bigoplus_{\sigma} R_\sigma$, that clearly has $\text{rk}(G) = 2^{\aleph_0}$. Then $\text{End}(G) = \prod_{\sigma} \mathbb{Z}$, and for every endomorphism ϕ of G each subgroup R_σ is ϕ -invariant. Therefore $\phi : G \rightarrow G$ satisfies the conditions of Corollary 5.11, with $G_\sigma = \bigoplus_{\alpha \leq \sigma} R_\alpha$, so $\widetilde{\text{ent}}(\phi) = 0$. Hence $\widetilde{\text{ent}}(G) = 0$.

In the case of torsion Abelian groups G , Proposition 3.6 gives $\text{ent}(G) = \widetilde{\text{ent}}(G)$. Many examples of torsion Abelian groups G with $\text{ent}(G) = 0$ can be found in [9]. Nevertheless, the problem of the classification of all torsion Abelian groups with this property is open (see [9, 19]).

6 The Uniqueness Theorem

In this section $h^* : \text{Mod}(\mathbb{Z}[X]) \rightarrow \mathbb{R}^*$ denotes a length function and, given $G_\phi \in \text{Mod}(\mathbb{Z}[X])$, we will often write $h^*(\phi)$ in place of $h^*(G_\phi)$. We characterize $\widetilde{\text{ent}}$ as the unique length function of $\text{Mod}(\mathbb{Z}[X])$ satisfying two natural hypotheses. Indeed, we prove the following

Theorem 6.1 (Uniqueness Theorem). *There exists a unique length function $h^* : \text{Mod}(\mathbb{Z}[X]) \rightarrow \mathbb{R}^*$ such that*

- (a) $h^*(\beta_K) = \log |K|$ for any finite Abelian group K , where $\beta_K : K^{(\mathbb{N})} \rightarrow K^{(\mathbb{N})}$ is the right Bernoulli shift;
- (b) $h^*(\phi) = \log s$ for every automorphism ϕ of \mathbb{Q}^m whose characteristic polynomial over \mathbb{Q} has the positive integer s as least common multiple of the denominators of its coefficients.

Furthermore, $h^* = \widetilde{\text{ent}}$.

The proof of the Uniqueness Theorem will follow at the end of this section after three lemmas describing some properties of the length functions of $\text{Mod}(\mathbb{Z}[X])$. Note that an analogous result has been proved for the algebraic entropy h in [6], with hypothesis (b) replaced by the Algebraic Yuzvinski Formula:

- (b') $h(\phi) = \mathfrak{m}(\phi)$ for every automorphism ϕ of \mathbb{Q}^m whose minimal polynomial over \mathbb{Q} has the positive integer s as least common multiple of the denominators of its coefficients.

Furthermore, it was proved in [9] that ent is the unique length function defined on the endomorphisms of torsion Abelian groups satisfying hypothesis (a). We remark that in [9] also the logarithmic law appeared in the hypothesis of such theorem, anyway one can use the following Lemma 6.2 to see that this is not necessary.

The next lemma proves that condition (a) in the statement of the Uniqueness Theorem implies that h^* is trivial on all the endomorphisms of finite Abelian groups. The argument used in the proof already appeared in [19].

Lemma 6.2. *Let $h^* : \text{Mod}(\mathbb{Z}[X]) \rightarrow \mathbb{R}^*$ be a length function satisfying the hypothesis (a) of the Uniqueness Theorem. Then $h^*(\phi) = 0$ for all $\phi : G \rightarrow G$, with G a finite Abelian group.*

Proof. Consider the surjective homomorphism $f : G^{(\mathbb{N})} \rightarrow G$ such that $f((x_n)_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} \phi^n(x_n)$; note also that $f\beta_G = \phi f$, and so f is a surjective homomorphism of $\mathbb{Z}[X]$ -modules $f : G_{\beta_G}^{(\mathbb{N})} \rightarrow G_\phi$. Thus, by the additivity of h^* ,

$$h^*(\phi) = h^*(\beta_G) - h^*(\ker(f)_{\beta_G}). \quad (6.1)$$

On the other hand, consider the injective homomorphism $g : G^{(\mathbb{N})} \rightarrow G^{(\mathbb{N})}$ defined on the generators as follows

$$g(0, \dots, 0, \underbrace{x}_n, 0, \dots) = (0, \dots, 0, \underbrace{\phi(x)}_n, -x, 0, \dots).$$

Note that $g\beta_G = \beta_G g$ and $fg = 0$; hence g is an embedding of $\mathbb{Z}[X]$ -modules of $G_{\beta_G}^{(\mathbb{N})}$ in $\ker(f)_{\beta_G}$ and so $h^*(\beta_G) \leq h^*(\ker(f)_{\beta_G}) \leq h^*(\beta_G)$. By (6.1), $h^*(\phi) = 0$. \square

In the next lemma we prove that any length function h^* of $\text{Mod}(\mathbb{Z}[X])$ which satisfies condition (a) in the statement of the Uniqueness Theorem, necessarily coincides with $\widetilde{\text{ent}}$ on finitely generated $\mathbb{Z}[X]$ -modules G_ϕ such that $\text{rk}(G) = \infty$ (see also Proposition 5.6).

Lemma 6.3. *Let $h^* : \text{Mod}(\mathbb{Z}[X]) \rightarrow \mathbb{R}^*$ be a length function satisfying the hypothesis (a) of the Uniqueness Theorem. Let G be an Abelian group of infinite rank, $\phi : G \rightarrow G$ an endomorphism and F a finitely generated subgroup of G such that $G = T(\phi, F)$. Then $h^*(\phi) = \infty$.*

Proof. By Proposition 5.6 there is a ϕ -invariant subgroup $H \cong \mathbb{Z}^{(\mathbb{N})}$ of G on which the restriction of ϕ acts as the right Bernoulli shift $\beta_{\mathbb{Z}}$. Thus, it is enough to show that $h^*(H_\phi) = \infty$. This follows by the fact that $(H/pH)_\phi \cong \mathbb{Z}(p)_{\beta_{\mathbb{Z}(p)}}^{(\mathbb{N})}$ and so $h^*(H_\phi) \geq \sup\{h^*(\beta_{\mathbb{Z}(p)}) : p \in \mathbb{P}\} = \sup\{\log(p) : p \in \mathbb{P}\} = \infty$. \square

The next lemma is the last result that we need for the proof of the Uniqueness Theorem, the proof is exactly the same of Corollary 3.15 and so it is omitted.

Lemma 6.4. *Let $h^* : \text{Mod}(\mathbb{Z}[X]) \rightarrow \mathbb{R}^*$ be an upper continuous invariant. Let G be a torsion-free Abelian group and $\phi : G \rightarrow G$ an endomorphism. Then $h^*(\phi) = h^*(\widetilde{\phi})$, where $\widetilde{\phi} : D(G) \rightarrow D(G)$ is the unique extension of ϕ to the divisible hull $D(G)$ of G .*

Proof of the Uniqueness Theorem 6.1. We already noted that $\widetilde{\text{ent}}$ is an additive upper continuous invariant; it satisfies (a) since over torsion Abelian groups it coincides with ent by Proposition 3.6 and because of [9, Example 1.9]; finally, it satisfies (b) in view of Theorem 4.2.

Conversely, let h^* be a length function on $\text{Mod}(\mathbb{Z}[X])$ satisfying (a) and (b). We have to show that h^* coincides with $\widetilde{\text{ent}}$. It suffices to verify that these two length functions coincide on finitely generated $\mathbb{Z}[X]$ -modules. So assume without loss of generality that $G = T(\phi, F)$ for some finitely generated subgroup F of G .

If G is torsion, then G is either finite or there exists an element $x \in G[p]$ for some prime p such that $T(\phi, x\mathbb{Z}) \cong \mathbb{Z}(p)^{(\mathbb{N})}$ and ϕ acts on this subgroup as the right Bernoulli shift $\beta_{\mathbb{Z}(p)}$. So, using an inductive argument, one can realize the group G as the union of a finite chain of ϕ -invariant subgroups

$$0 = (H_0)_\phi \subseteq (H_1)_\phi \subseteq \dots \subseteq (H_n)_\phi = G_\phi$$

such that $(H_{i+1}/H_i)_\phi$ is either finite or conjugated to $\mathbb{Z}(p_i)_{\beta_{\mathbb{Z}(p_i)}}^{(\mathbb{N})}$, for all $i < n$ and suitable primes p_i . Hence, $\widetilde{\text{ent}}(\phi) = h^*(\phi)$ by the additivity of both invariants, condition (a) and Lemma 6.2.

If G is torsion-free, then $\widetilde{\text{ent}}(\phi) = h^*(\phi)$ by (b) and Lemma 6.4 in case $\text{rk}(G) < \infty$, by Lemma 6.3 and Proposition 5.6 in case $\text{rk}(G) = \infty$.

The general case follows by the torsion and torsion-free cases using additivity. \square

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